CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Lines and Linear Equations

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Beta Version

Lines and Linear Equations

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to:

- Interpret speed as the slope of a linear graph.
- Translate between the equation of a line and its graphical representation.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

8.EE: Understand the connections between proportional relationships, lines, and linear equations.

8.F: Define, evaluate and compare functions.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*:

2. Reason abstractly and quantitatively.
4. Model with mathematics.
7. Look for and make use of structure.

INTRODUCTION

The unit is structured in the following way:

- Before the lesson, students work individually on an assessment task that is designed to reveal their current understanding and difficulties. You then review their work and create questions for students to answer in order to improve their solutions.
- During the lesson, students work in small groups on a collaborative task, matching graphs, equations and pictures.
- Towards the end of the lesson there is a whole-class discussion.
- In a follow-up lesson, students review their initial solutions, and then use what they have learned to either revise the same introductory assessment task or complete a different task.

MATERIALS REQUIRED

- Each student will need copies of the assessment tasks, *The Race* and *The Race (revisited.)* It will help students if the two pages of each task are printed on separate pages.
- Each small group of students will need a mini-whiteboard, a pen, and an eraser, the cut up cards *Graphs 1*, *Graphs 2*, *Equations*, *Flowing Liquid,* a large sheet of paper for making a poster, and a glue stick. If you are using different colored paper, ensure cards *Graphs 1* and *Graphs 2* are printed on the same colored paper.
- There are some projector resources to support whole-class discussions.
- If possible, a real model of two identical containers connected at the neck to demonstrate the liquid flow. For example, two identical soda bottles, connected at the neck with tape.

TIME NEEDED

Approximately 20 minutes before the lesson, a 80-minute lesson (or two 40-minute lessons), and 20 minutes in a follow-up lesson or as homework. Exact timings will depend on the needs of the class.

BEFORE THE LESSON

Assessment task*: The Race* (20 minutes)

Have the students complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work, and find out the kinds of difficulties students have with it. You will then be able to target your help more effectively in the next lesson.

Give each student a copy of the assessment task *The Race*.

> *Read through the questions and try to answer them as carefully as you can.*

It is important that, as far as possible, students are allowed to answer the questions without your assistance.

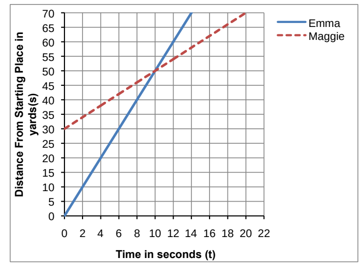
Students should not worry too much if they cannot understand or do everything because in the next lesson they will engage in a similar task, which should help them. Explain to students that, by the end of the next lesson, they should expect to answer questions such as these confidently. This is their goal.

The Race

Maggie and Emma race each other along a straight running track.

Maggie starts some distance ahead of Emma.

The graph describes the race.



1. After five seconds, who is running the fastest? Explain your answer.

....

....

2. Emma's line can be represented by the equation:

$s = 5t$ — s is the distance, in yards, from the Starting Place. t is the time, in seconds, from the start of the race.

What is the equation that represents Maggie's line?

....

3. Describe what happens in the race.

....

....

Assessing students' responses

Collect students' responses to the task and note down what their work reveals about their current levels of understanding, and their different approaches.

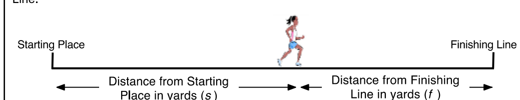
We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare their scores and distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a list of questions. Some suggestions for these are given in the *Common issues* table on the next page. We suggest that you make a list of your own questions, based on your students' work, using the ideas on the following page. We recommend you:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions, highlighting the questions relevant to individual students.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these on the board when you return the work to the students.

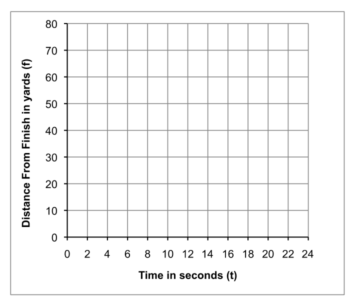
The diagram below shows the distance a runner is from the Starting Place and from the Finishing Line.



4. The following equation can also be used to describe Emma's race:

$f = -5t + 70$ — f is the distance, in yards, from the **Finishing Line**. t is the time, in seconds, from the beginning of the race.

a. Plot this line on the graph.



Emma
Maggie

On this graph the distance is measured from the runner to the finish, not the start.

b. Add a line to the graph that represents Maggie's race.

c. What is the equation of this second line?

....

Common issues:	**Suggested questions and prompts:**
Student assumes Maggie is running the fastest because at five seconds her line is above Emma's line (Q1.)	• During the race, does Emma's or Maggie's speed change? • How can you figure out the speed of each runner?
Students description of the race is limited For example: The student does not mention speed, or the time it took for each person to complete the race.	• What more can you tell me about the race? • Does one runner overtake the other one? If so, at what point does this happen? • Who wins the race? How far ahead are they when they cross the finishing line? • What are the race times for each runner?
Student misinterprets the scale For example: The student fails to notice the distance goes up in 5s not 1s (Q1.) Or: The student does not notice the scales for the axes on the two graphs are different.	• What is the scale on the vertical/horizontal axis for each graph?
Student draws an incorrect graph (Q4) For example: The student draws a graph with a positive slope. Or: The student draws a slope with an incorrect y-intercept, e.g. $f = 30$. Or: The student draws a non-linear graph. Or: The student draws an incomplete graph.	• As the race progresses will the distance, f, increase or decrease? How can you show this on your graph? • At the beginning of the race, how far are the runners from the finishing line? How can you show this on your graph? • Does Maggie run at a constant speed? How have you shown this speed on your graph? • Your graph should represent all of the race. When will Emma/Maggie have completed the race? How can you show these points on the graph?
Student's equations are incorrect For example: The student writes an equation without a variable for the time.	• Explain your equation in words. • Does your equation describe how the distance changes as the race progresses?

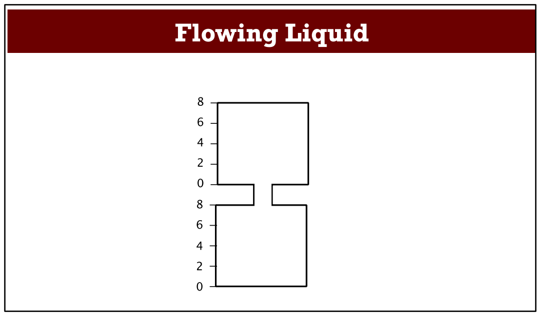
SUGGESTED LESSON OUTLINE

Whole-class introduction (20 minutes)

Give each student a mini-whiteboard, a pen, and an eraser.

If you have a real model of two identical containers connected at the neck then use it throughout the introduction to demonstrate the liquid flow. If there is liquid in the top container then, to ensure the smooth flow of liquid, there needs to be a hole in the base of this top container.

Show the class Slide P-1 of the projector resource.

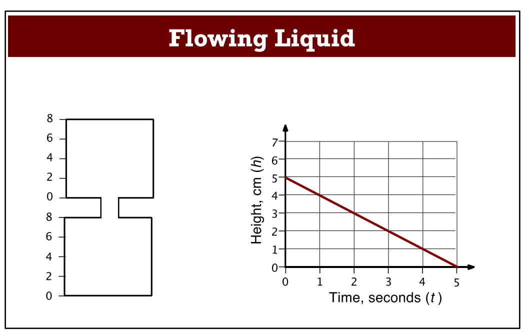


Explain to the class that the top and bottom containers are two identical right rectangular prisms. Liquid flows from the top to the bottom. The total height of liquid in both the containers is 6 units.

If the height of the liquid in the top right rectangular prism is 4 units, what is the height of the liquid in the bottom right rectangular prism? [2 units.]

At what height will there be equal amounts of liquid in the top and bottom prisms? [When there are 3 units of liquid in each prism.]

Now show Slide P-2 of the projector resource.



Ask students to describe in detail on their mini-whiteboards the flow of the liquid.

After a few minutes ask students to show you their whiteboards. Ask two or three students with different descriptions to explain them. Encourage the rest of the class to challenge, or add to, these descriptions.

To make sure students understand the context of the task, ask the following questions:

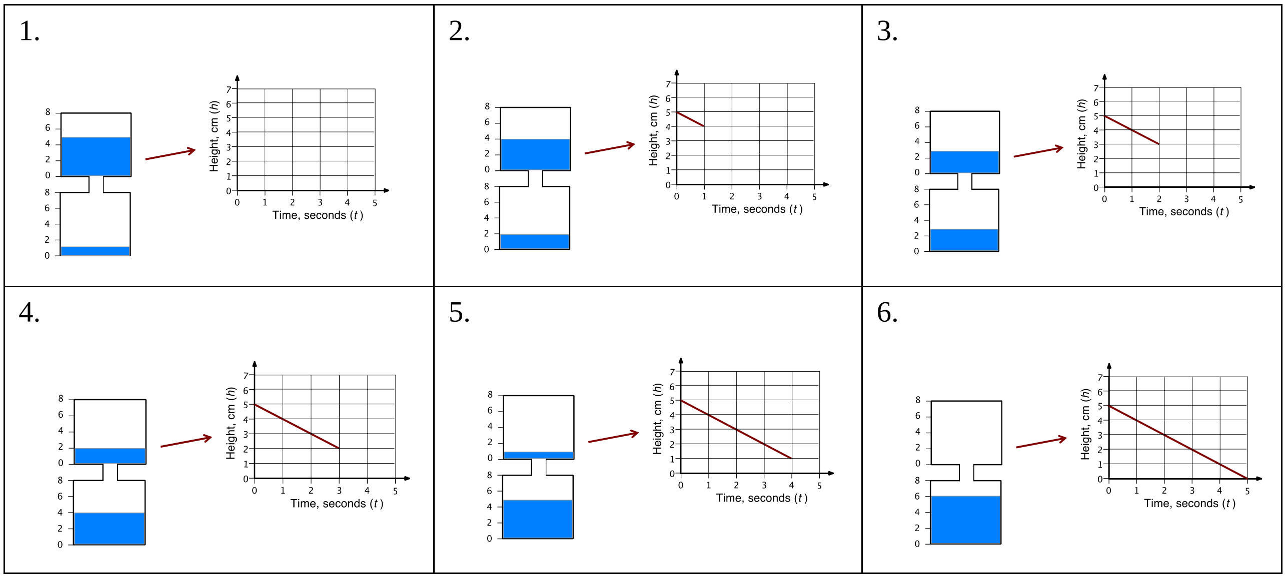
Does the graph show the flow of liquid out of the top or into the bottom prism? [Top.] How do you know?

What is the starting situation? [5 units of liquid in the top prism, 1 unit of liquid in the bottom prism.]

Does the liquid flow at a constant speed? [Yes.]
How do you know? [The slope is a straight line.]

What speed does the liquid flow at? [1 cm per second.]
How do you know?

Then show the sequence of slides P-3 to P-8. This visualization of the flow of liquid between the prisms should help students understand the context.

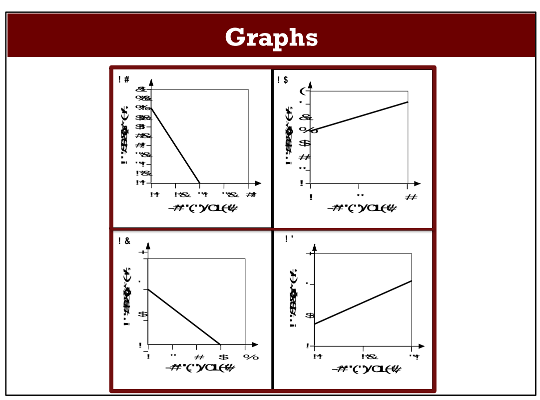


You may then want to ask further questions:

How can you change the starting situation so the liquid flows out in half the time? [Set the start height to 2.5 units, or double the speed of the flow of liquid, or increase the opening between the two prisms.]

How can you change the starting situation so the liquid flows out in double the time? [Halve the speed of the liquid or decrease the opening between the two prisms.]

Now show slide P-9.



Explain to students that they will be working in groups on some of these cards.

The graphs represent the flow of a liquid either out of the top prism or into the bottom prism of the container. Use the information from the graphs to figure out two graphs that represent the top and bottom prisms of the same container. [G2 and G6.]

This should allow students to absorb the context of the task individually, so that when they start to work in groups they **all** have something to contribute, not just the faster thinkers.

Collaborative activity 1: *Matching Graphs* (20 minutes)

Organize the class into groups of two or three students. Give each group the cards *Graphs 1* and *2*. Explain how students are to work collaboratively.

The graphs I've given out represent the flow of a liquid either out of the top prism or into the bottom prism of the container.

Take it in turns to match two cards that represent the movement of liquid in one container.

Place them next to each other, not on top so that everyone can see.

When you match two cards, explain how you came to your decision.

Your partner should either explain that reasoning again in his or her own words, or challenge the reasons you gave.

You both need to be able to agree on and explain the match of every card.

Some graphs are missing information, such as a scale along an axis. You will need to add this scale.

Slide P-10, *Working Together*, summarizes how students should work together.

The purpose of this structured group work is to make students engage with each other's explanations, and take responsibility for each other's understanding.

If some students are finding this matching difficult then give them the cards *Flowing Liquid*. Students are to match one of these cards with two of the *Graph* cards.

While students work in small groups move around the class, noting different student approaches to the task and supporting student reasoning.

Note different student approaches to the task

Notice how students make a start on the task, where they get stuck, and how they respond if they do come to a halt. Do students assume that two matched *Graph* cards must have the same vertical intercept? Do students pay attention to the scale? Are students figuring out the slope and if so do they use ratios or fractions? Do students look at multiple attributes of each graph? You can use this information to focus a whole-class discussion towards the end of the lesson.

Support student reasoning

Try not to make suggestions that move students towards a particular placement. Instead, ask questions to help students to reason together.

If students get stuck you may want to ask:

Can you think of one specific question you want to ask?

This question requires students to think carefully about the task, and in so doing may help them to get started. Either answer the question yourself, or ask a member of the group to answer the question.

State one thing this graph tells you about the flow of the liquid. Now tell me another.

What is the start height of the liquid? What must the start height for its connecting prism be?

How many seconds is the liquid flowing? How many seconds must the liquid be flowing in the connecting prism?

How is the speed of flow of liquid represented in this graph? What is it?

To ensure students are explaining their reasoning to one another, you may want to ask:

Amy matched these two cards. Andrew, why does Amy think these two cards go together?

If you find the student is unable to answer that question, ask them to discuss the matching further. Explain to the group that you will return in a few minutes to ask a similar question.

Taking two lessons to complete all activities

If you decide to extend the lesson over two periods then 5 minutes before the end of the first lesson ask students to note down their existing card matches, and then paper clip all their cards together.

At the start of the second lesson spend some time reminding the class about the activities.

Sharing Work (10 minutes)

As students finish matching the cards, ask one student from each group to visit another group's desk.

If you are staying at your desk, be ready to explain the reasons for your group's graph matches.

If you are visiting another group, copy your matches onto a piece of paper.

Go to another group's desk and check to see which matches are different from your own. If there are differences, ask for an explanation. If you still don't agree, explain your own thinking.

When you return to your own desk, you need to consider as a group whether to make any changes to your own work.

You may want to use Slide P-11 of the projector resource, *Sharing Work*, to display these instructions.

Collaborative activity 2: *Matching Equations* and *Prisms* (15 minutes)

As groups complete the *Sharing Work* activity give them the *Equations* cards.

These cards represent algebraically the flow of liquid.

You are now to match each of these cards with the cards already on your desk.

If there is no equation card for your matches, make one up!

Again encourage students to spend some time thinking about how they intend to complete the task.

Support the students as in the first collaborative activity.

For this equation, before liquid starts to flow, what is the height of the liquid? How do you know?

Does this equation represent the top or bottom prism? How do you know?

How is the speed of flow represented in this equation?

As students finish the matching give to each group the *Flowing Liquid* cards.

These cards show the situation of the prisms before water has started to flow from the top prism to the bottom one. Students should add any missing information to the cards.

As students finish the activity give them a large sheet of paper for making a poster and a glue stick. They are to glue all the cards onto the paper and then attach the poster to the classroom wall for all to see.

Whole-class discussion (15 minutes)

Organize a discussion about what has been learned.

Depending on how the lesson went, you may want to first focus on the common mistakes students made, review what has been learnt and what they are still struggling with, and then extend and generalize the math.

Use what you have noticed about the way students have worked to select one or two groups to explain their approach.

How did you decide that this equation matched these graphs/ this picture?

How did you decide what to add to this card?

Does anyone have any questions about this method?

Did anyone use a different/ similar method?

If you have time to extend the math, write the equation below on the board:

$$h = 5t + 1$$

Ask the following questions in turn.

This equation describes the flow of liquid in one of the prisms of the container.

On your whiteboards write an equation that describes the flow of the liquid in the other prism of the same container. [$h = -5t + 5$.]

On your whiteboards write an equation that describes the flow of a liquid in this prism that takes half the time. [$h = -10t + 5$ or $h = -5t + 2.5$, when $h = 0$, $t = 0.5$]

On your whiteboards write an equation that describes the flow of liquid that takes one second longer. [$h = -2.5t + 5$, when $h = 0$ $t = 2$.]

Ask two or thee students with different equations to explain them. Encourage the rest of the class to challenge their answers.

Make up your own equation. Describe to your neighbor how the flow of liquid represented by this equation compares to the flow described by the equation on the board.

Follow-up lesson: Reviewing the assessment task (20 minutes)

Return to the students their response to the original assessment task.

If students struggled with the original assessment they may benefit from revising this assessment. In order that students can see their own progress, ask them to complete the task using a different color pen. Otherwise give students a copy of the task *The Race (revisited.)*

To connect the lesson activity with the assessment you may first want to ask students:

What do the two measurements for the distance run, s and f, have in common with the measurement of the liquid in the two prisms of each container? [As one measurement increases the other decreases at the same rate. In the race, as the distance from the start increases, then the distance from the finishing line decreases at the same rate. The total distance, $s + f$ is constant (70) throughout the race. In the container, as the liquid in the bottom prism increases then the liquid in the top one decreases at the same rate. The total liquid in the top and bottom container is constant (6).]

Ask students to look again at their original, individual, solutions to the problems together with your comments. If you have not added questions to individual pieces of work then write your list of questions on the board. Students should select from this list only those questions they think are appropriate to their own work. If you are short of time, you could set this task in the next lesson or for homework.

Read through your original solutions to The Race *problems.*

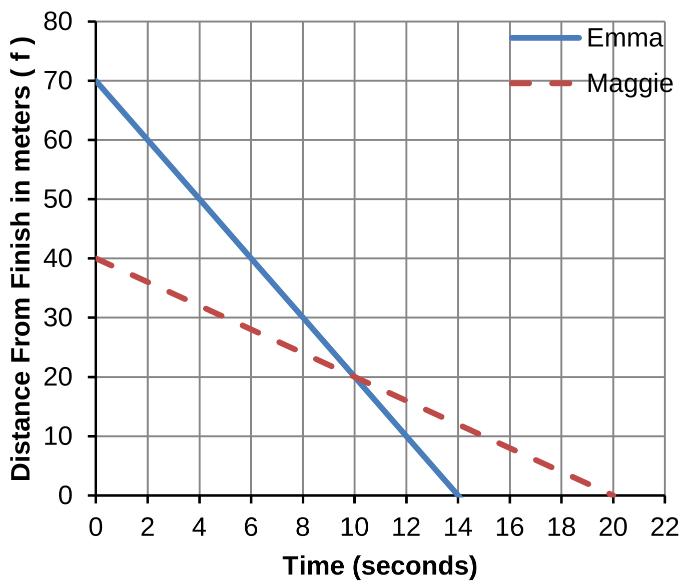
Make some notes on what you have learned during the lesson.

Use what you have learned to complete the new assessment task/revise your answers.

SOLUTIONS

Assessment Task: *The Race*

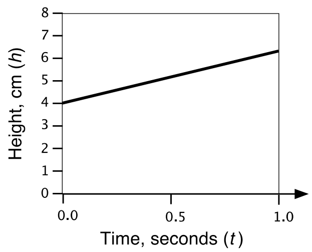
1. Emma. The slope of Emma's line is greater than Maggie's line.
2. $s = 2t + 30$.
3. Maggie starts the race 30 meters ahead of Emma. Emma runs 70 meters, Maggie runs 40 meters. Maggie runs at the constant speed of 2 meters per second. Emma runs at the constant speed of 5 meters per second. After 10 seconds Emma overtakes Maggie. Emma completes the race in 14 seconds, Maggie completes it in 20 seconds.
4. a.



b. $f = -2t + 40$

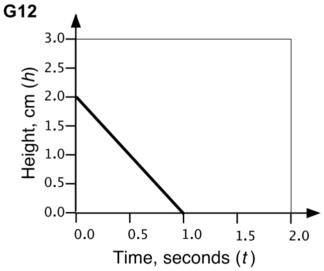
Collaborative activities

G1



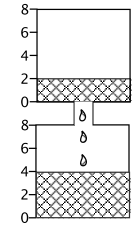
E12 $h = 2t + 4$

G12

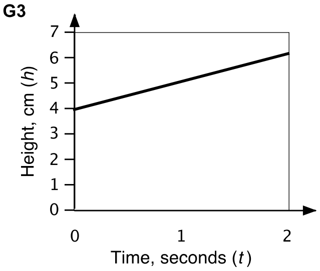


E5 $h = -2t + 2$

F6 Change in height: 2 cm per second

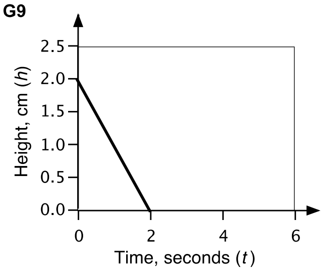


G3



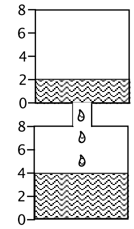
E3 $h = t + 4$

G9

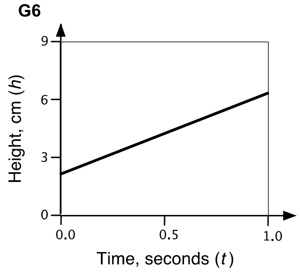


E9 $h = -t + 2$

F3 Change in height: 1 cm per second

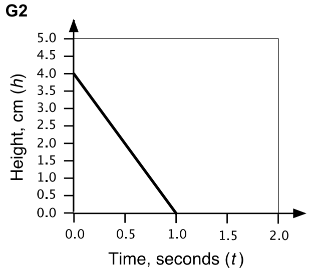


G6



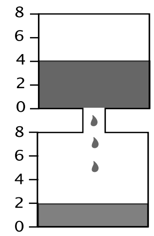
E4 $h = 4t + 2$

G2

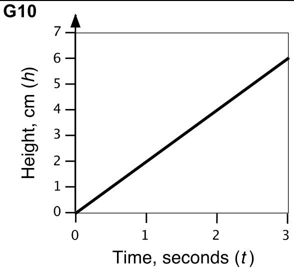


E6 $h = -4t + 4$

F4 Change in height: 4 cm per second

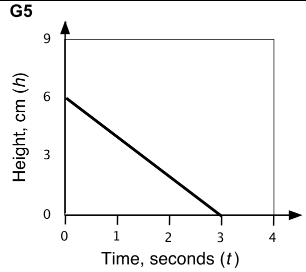


G10



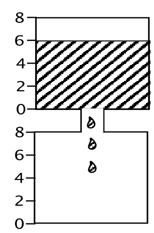
E2 $h = 2t$

G5

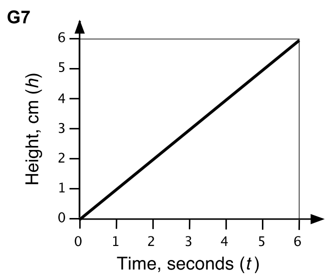


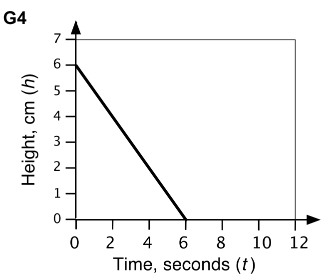
E8 $h = -2t + 6$

F5 Change in height: 2 cm per second

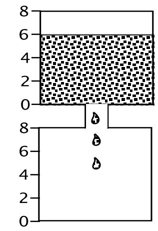


Collaborative activities (continued)



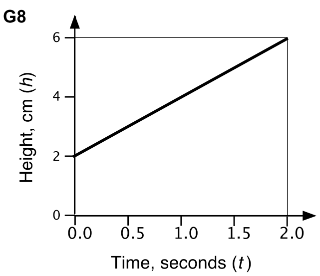


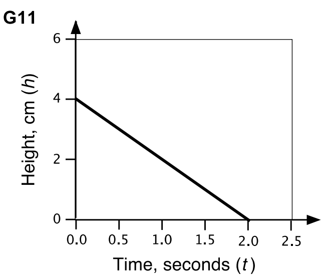
F1 Change in height: 1 cm per second



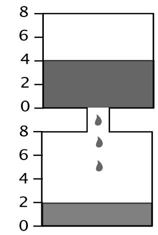
E7 $h = t$

E1 $h = -t + 6$





F2 Change in height: 2 cm per second

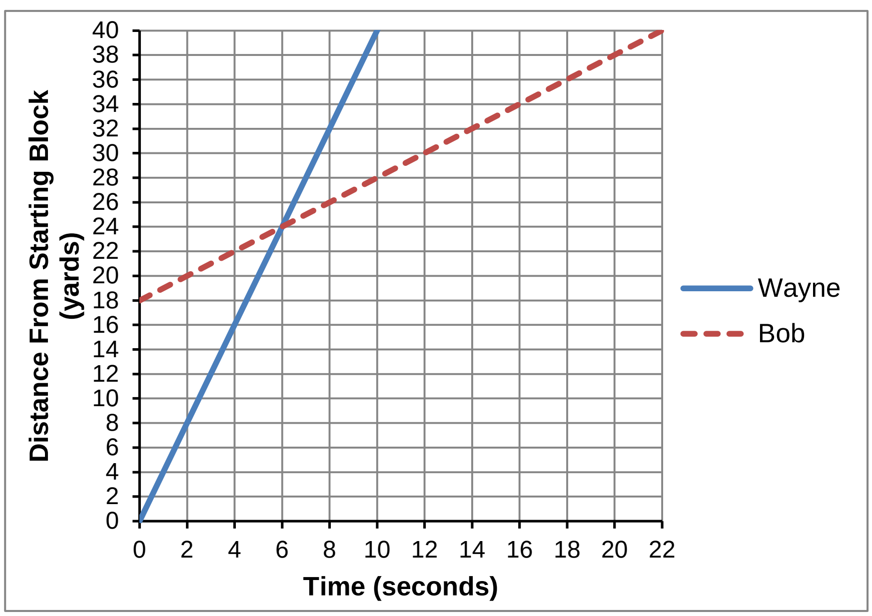


E10 $h = 2t + 2$

E11 $h = -2t$ -+ 4

Assessment Task: *The Race (revisited)*

1. a & b.



c. $s = t + 18$.

2. Wayne is always running faster than Bob. The slope representing his race is always steeper than the slope representing Wayne's race.

3. Equation of Wayne's line: $f = -4t + 40$. Equation of Bob's line: $f = -t + 22$.

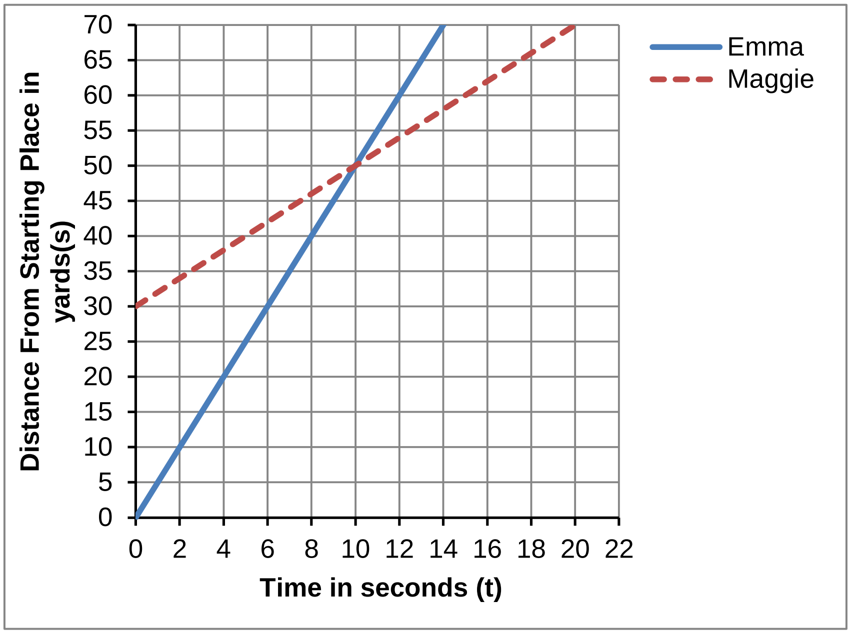
4. Wayne gets to the finishing line first. When $f = 0$ (at the finishing line) $t = 10$ for Wayne, but $t = 22$ for Bob. This means Wayne finishes the race12 seconds before Bob.

The Race

Maggie and Emma race each other along a straight running track.

Maggie starts some distance ahead of Emma.

The graph describes the race.



1. After 5 seconds, who is running the fastest? Explain your answer.

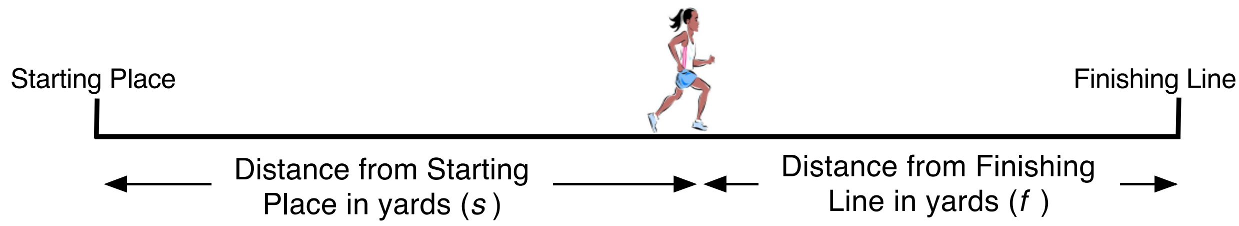
2. Emma's line can be represented by the equation:

$s = 5t$ — s is the distance, in yards, from the Starting Place.
t is the time, in seconds, from the start of the race.

What is the equation that represents Maggie's line?

3. Describe what happens in the race.

The diagram below shows the distance a runner is from the Starting Place and from the Finishing Line.



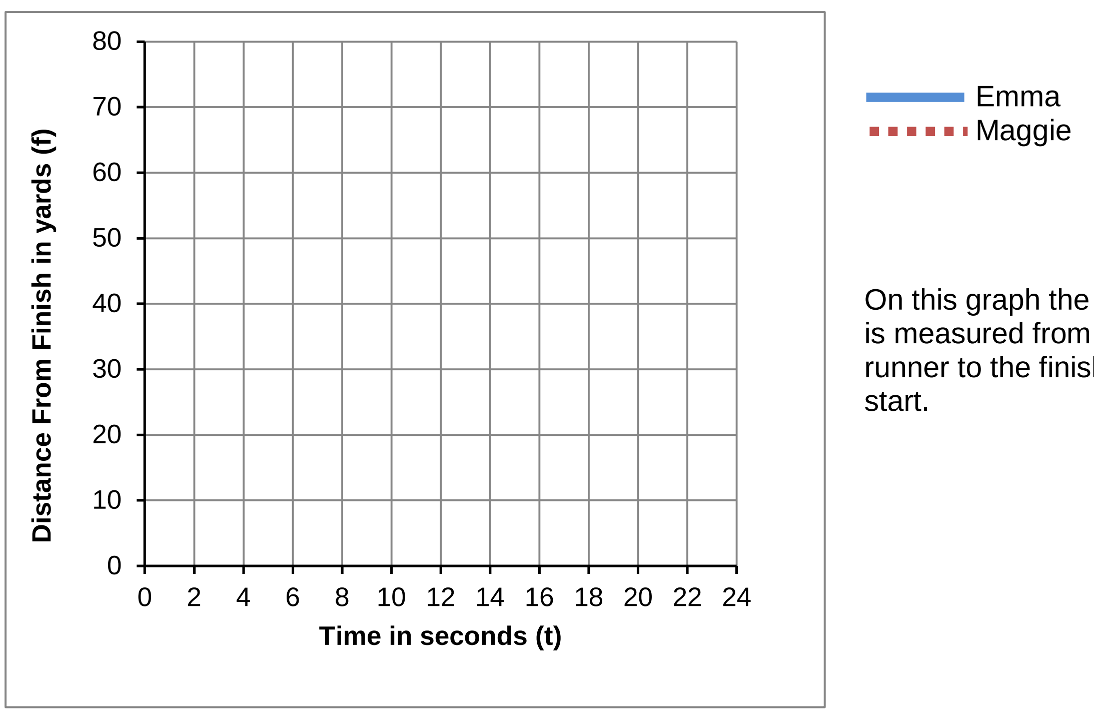
4. The following equation can also be used to describe Emma's race:

$f = -5t + 70$

f is the distance, in yards, from the **Finishing Line**.

t is the time, in seconds, from the beginning of the race.

a. Plot this line on the graph.



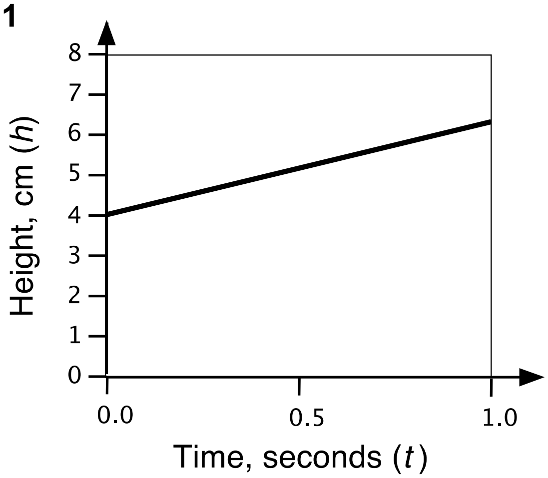
On this graph the distance is measured from the runner to the finish, not the start.

b. Add a line to the graph that represents Maggie's race.

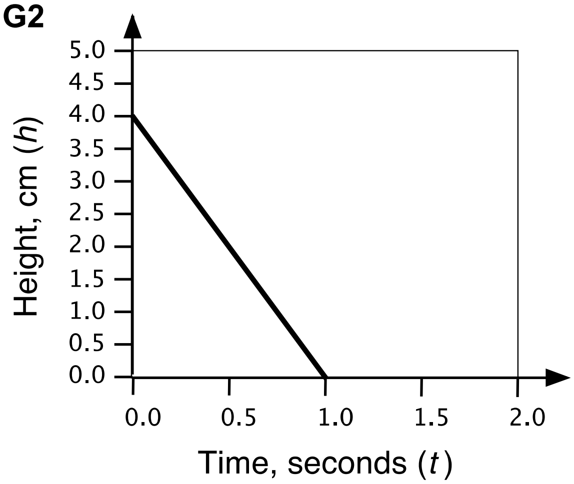
c. What is the equation of this second line?

Cards: Graphs 1

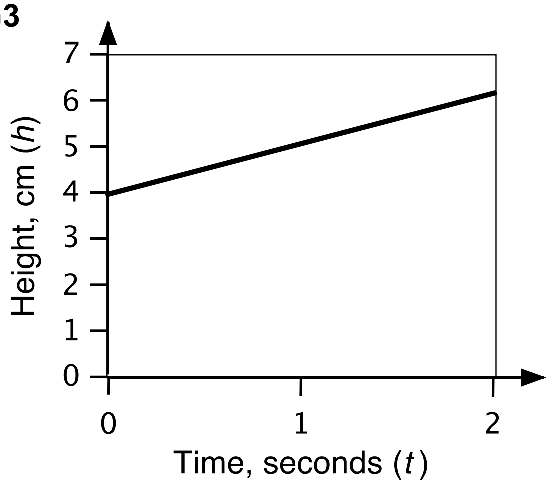
G1



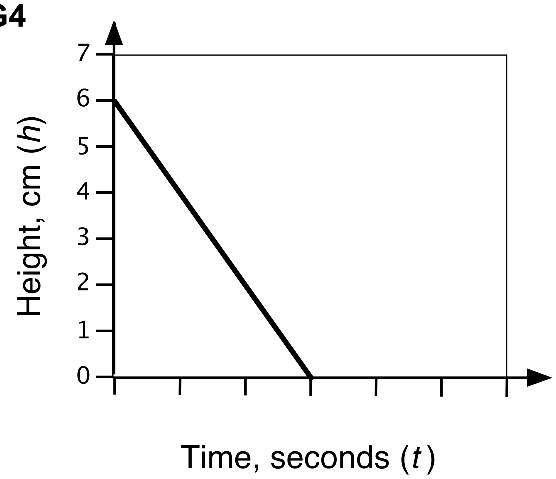
G2



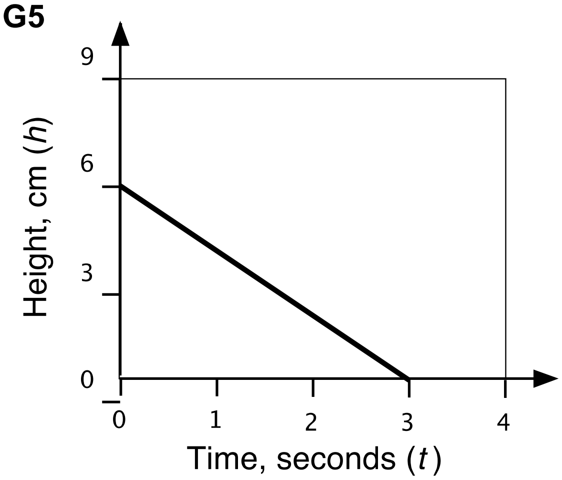
G3



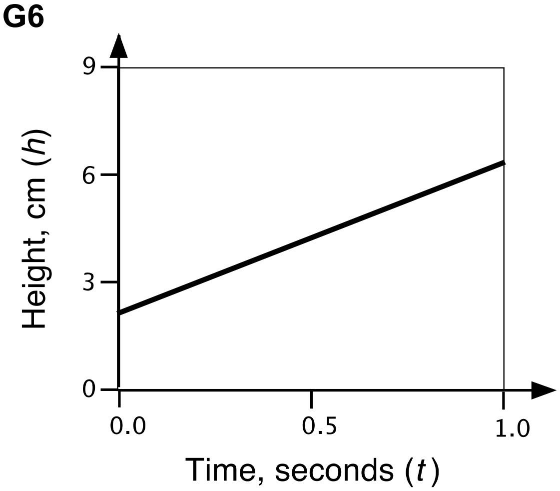
G4



G5

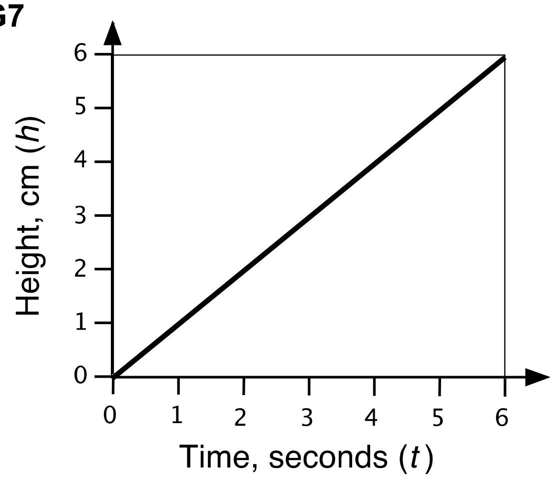


G6

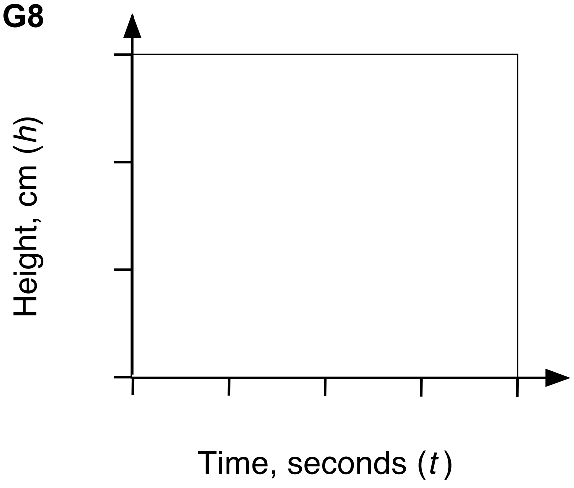


Cards: Graphs 2

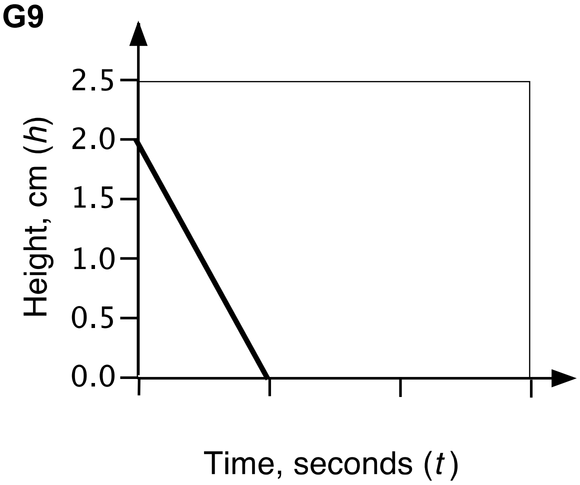
G7



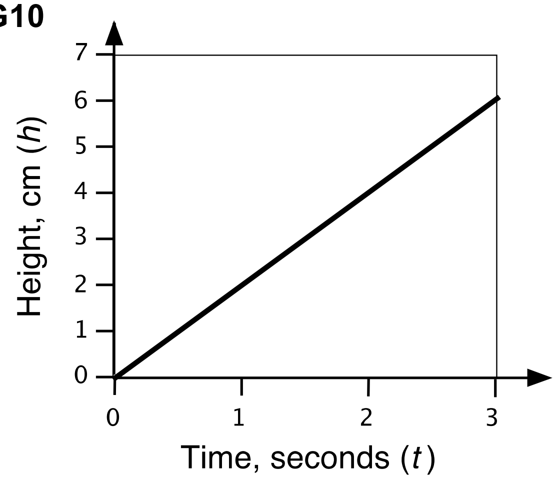
G8



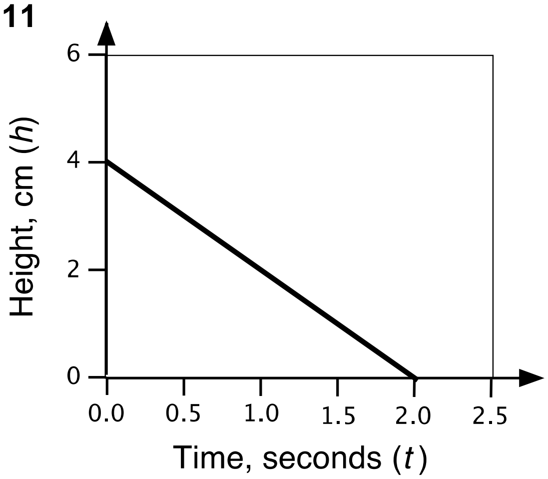
G9



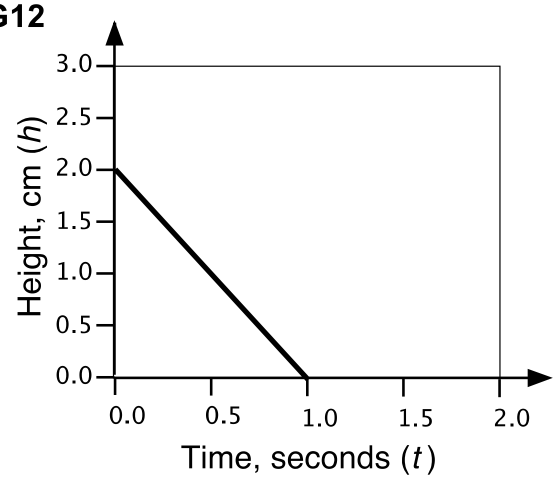
G10



G11



G12

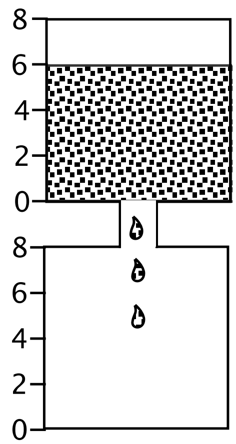


Cards: Equations

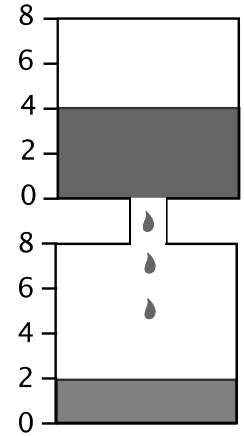
E1 $h = -t + 6$	E2 $h = -2t + 6$
E3 $h = t + 4$	E4 $h = 4t + 2$
E5 $h = -2t + 2$	E6 $h = -4t + 4$
E7 $h = t$	E8 $h = 2t$
E9 $h = -t + 2$	E10 $h = 2t + 2$
E11 $h = -2t + 4$	E12

Cards: Flowing Liquid

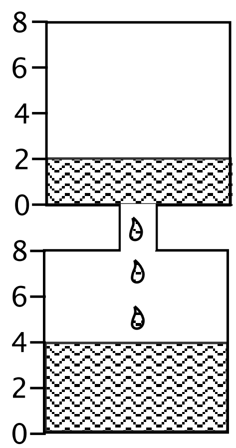
F1 Change in height: 1 cm per second



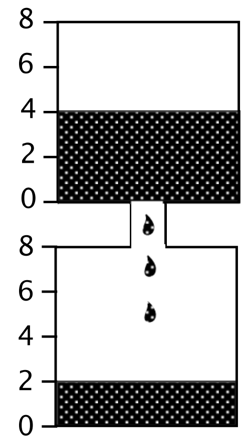
F2 Change in height: 2 cm per second



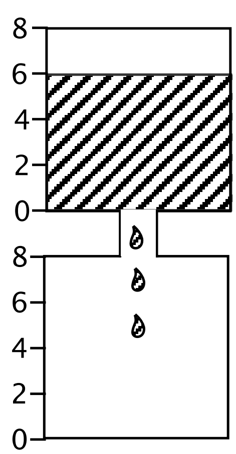
F3 Change in height: 1 cm per second



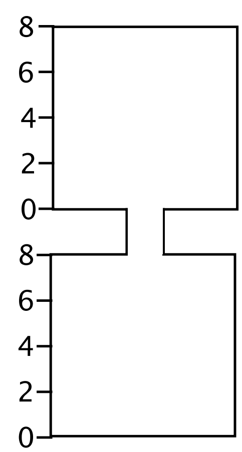
F4 Change in height: _ _ _ cm per second



F5 Change in height: _ _ _ cm per second



F6 Change in height: 2 cm per second



The Race (revisited)

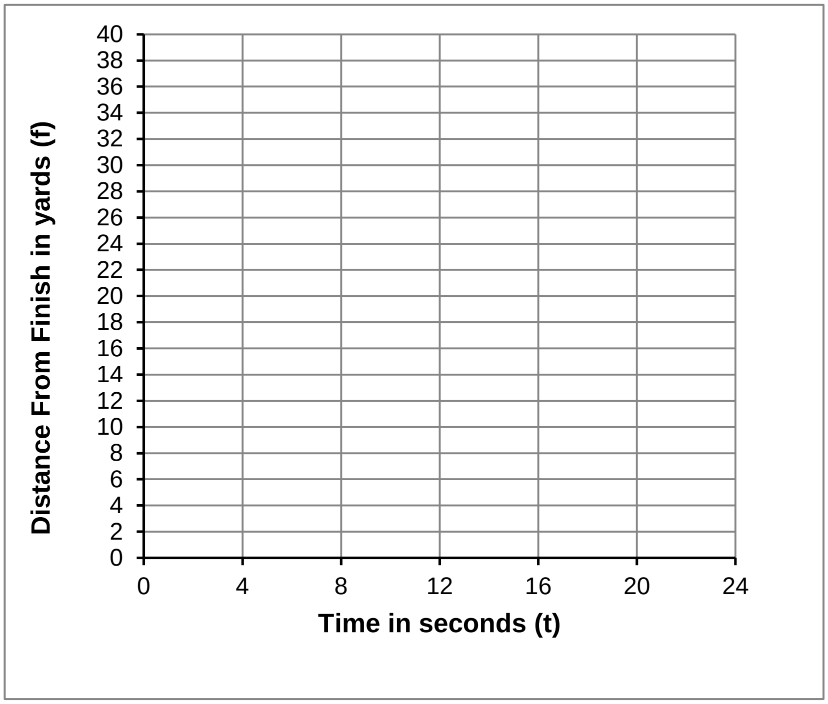
Wayne and Bob race each other along a straight running track.

1. The following equation can be used to describe Wayne's race:

$s = 4t$ s is the distance, in yards, from the **Starting Line**.

t is the time, in seconds, from the beginning of the race.

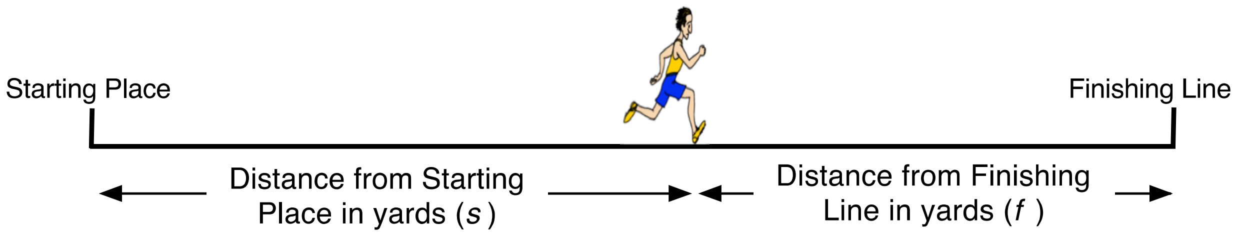
a. Plot this line on the graph.



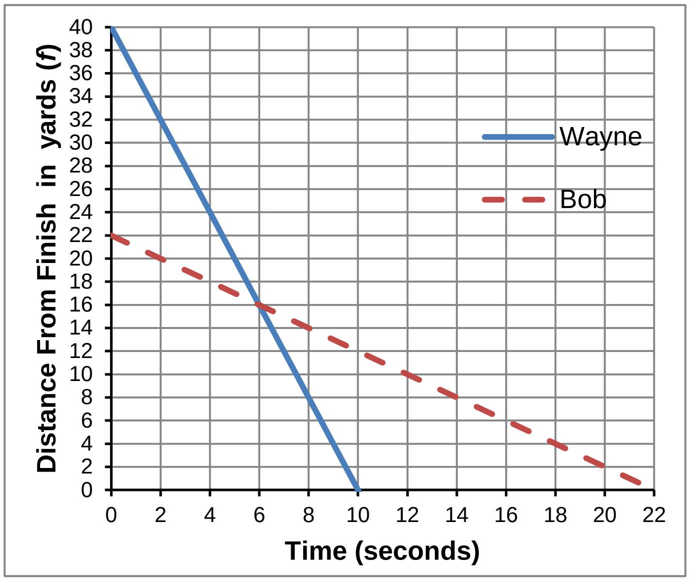
b. Bob starts 18 yard ahead of Wayne. He runs at a speed of 1 yard per second.
Pot a second line on the graph that represents Bob's race.

c. What is the equation of this second line?

The diagram below shows the distance a runner is from the Starting Place and from the Finishing Line.



The graph describes the race.



On this graph the distance is measured from the runner to the finish, not the start.

2. When is one runner running faster than the other? Explain how you know.

3. If f is the distance, in yards, from the Finishing Place.
t is the time, in seconds, from the start of the race.

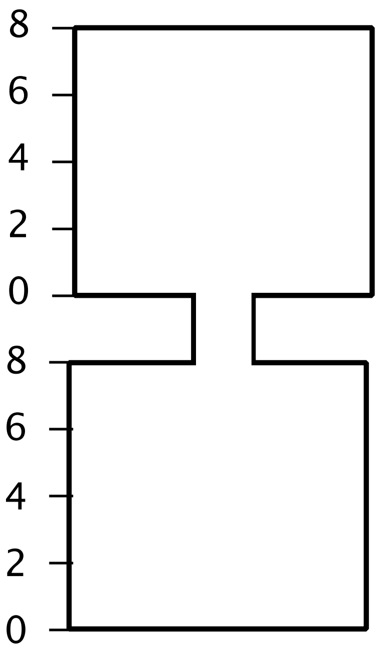
What are the equations of the two lines?

Equation of Wayne's line:

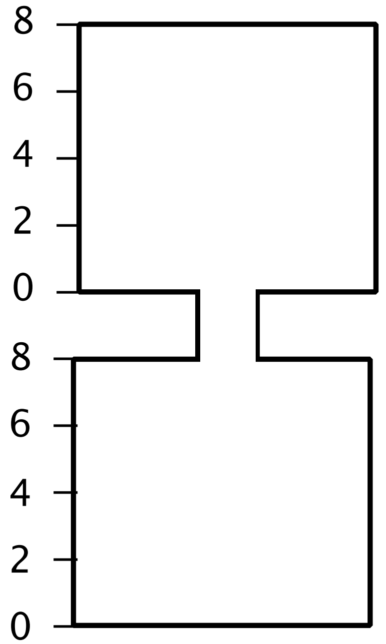
Equation of Bob's line:

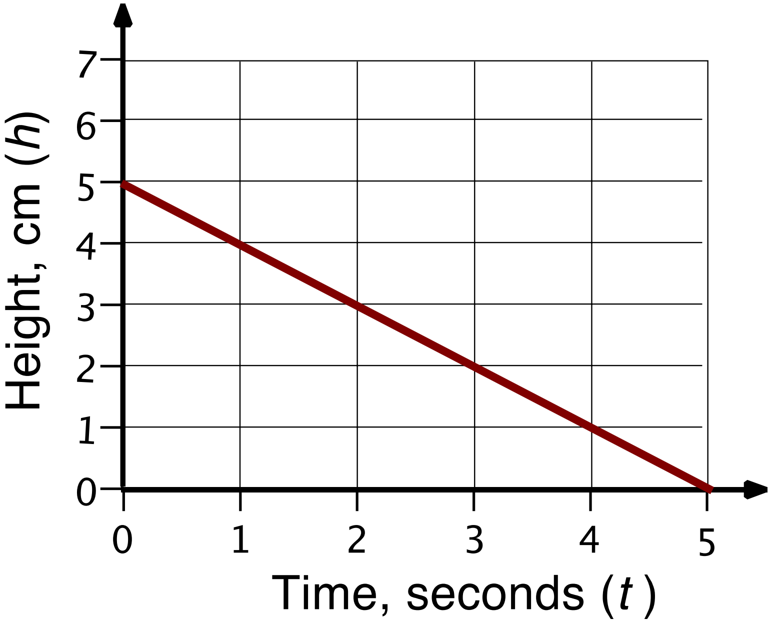
4. Who gets to the finishing line first? Explain how you know.

Flowing Liquid (1)

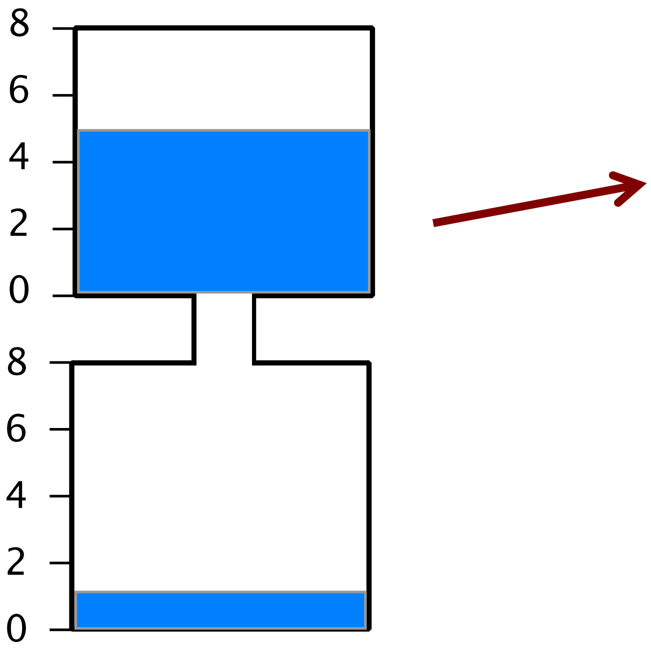


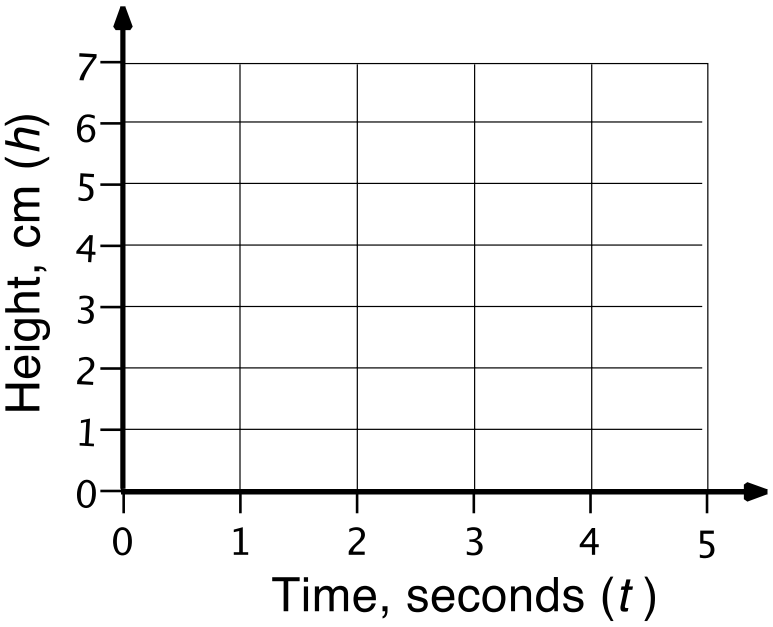
Flowing Liquid (2)



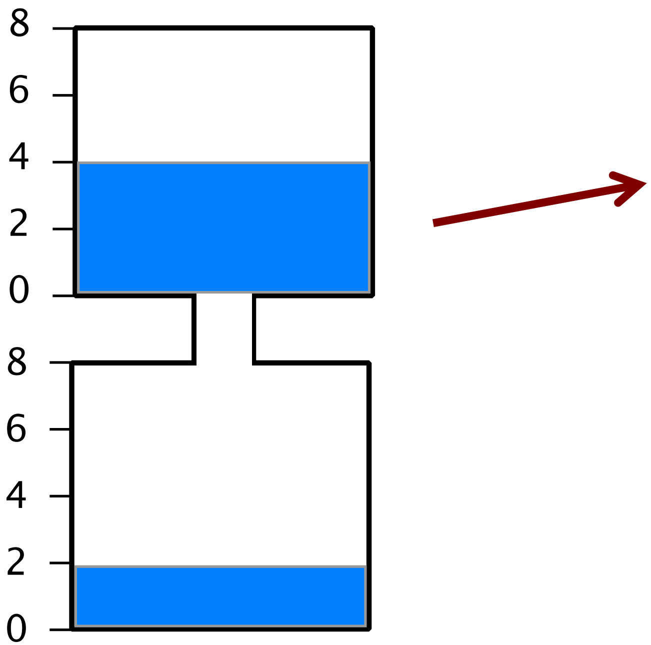


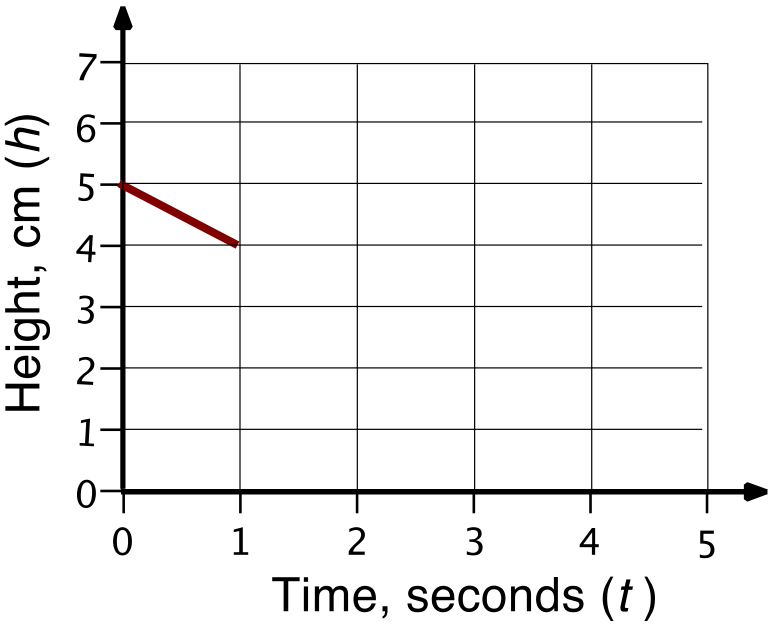
Liquid flowing out of the top prism 1



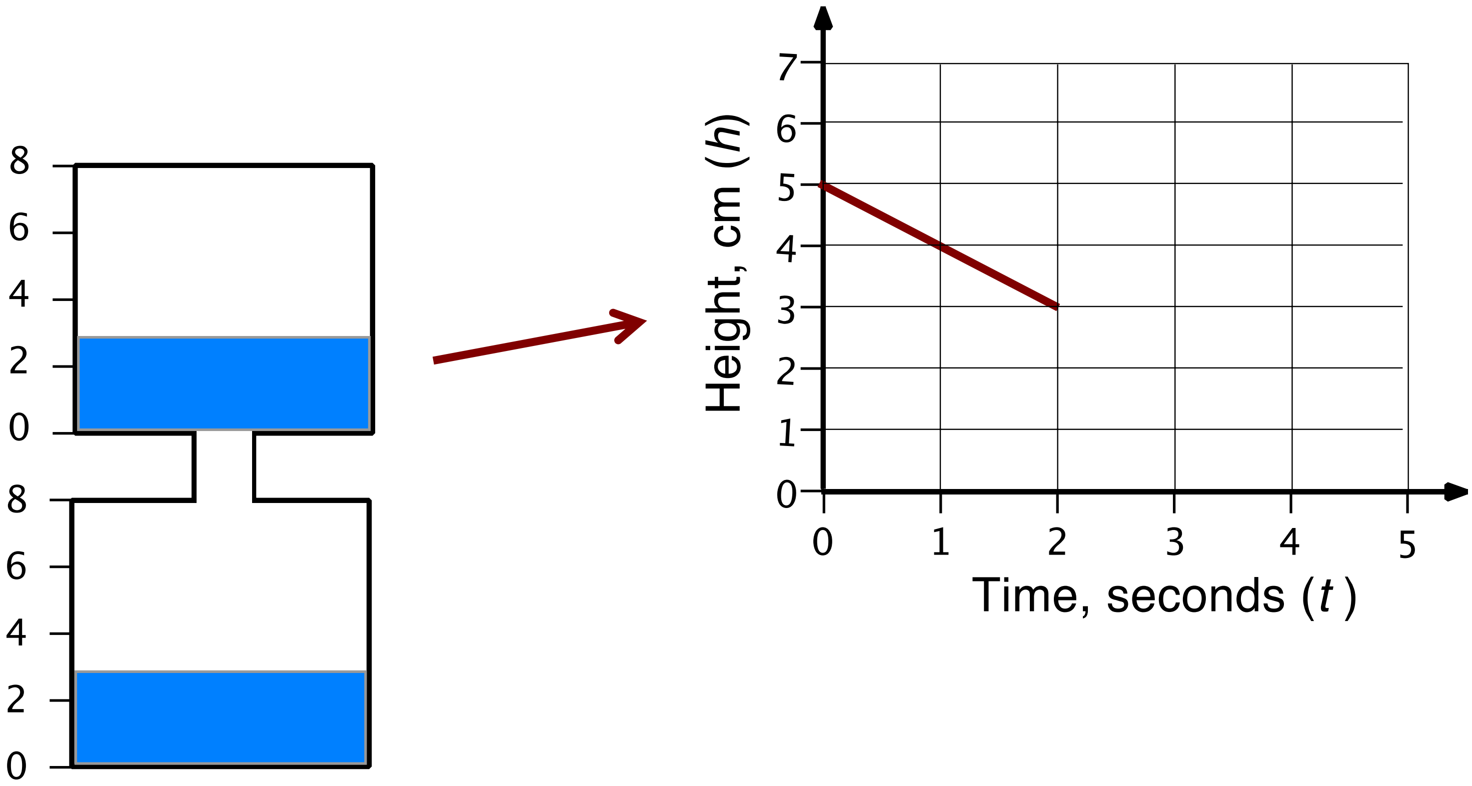


Liquid flowing out of the top prism 2

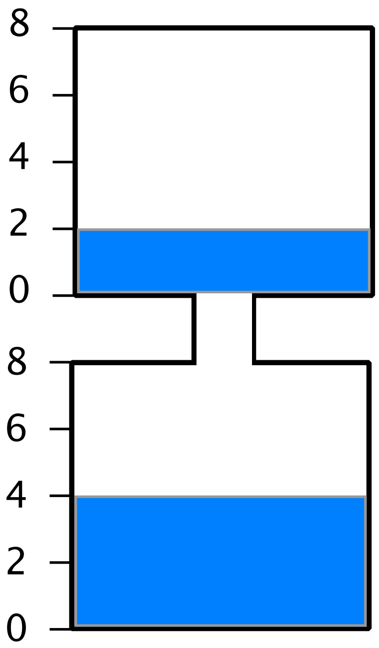


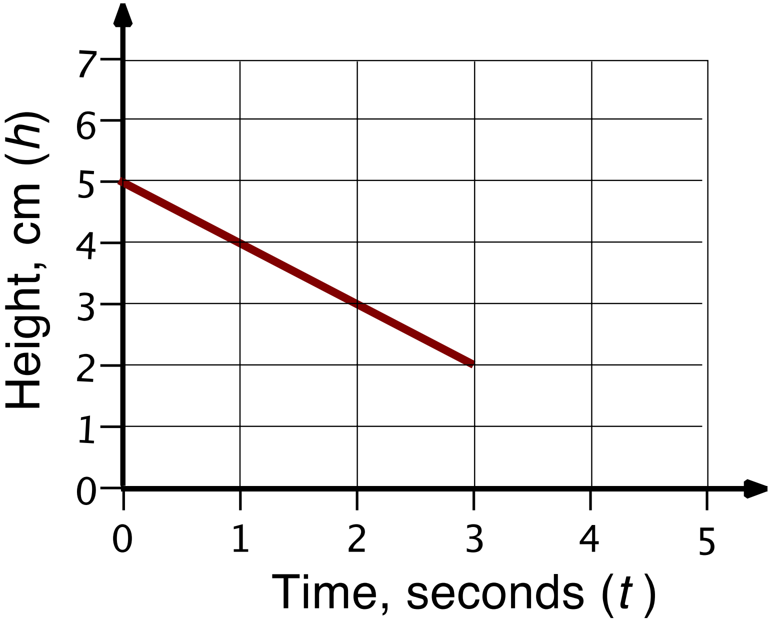


Liquid flowing out of the top prism 3

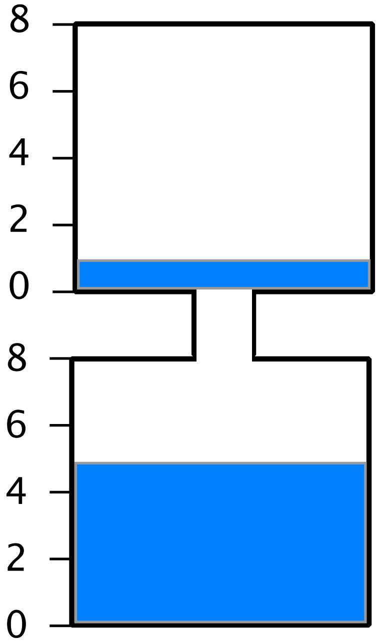


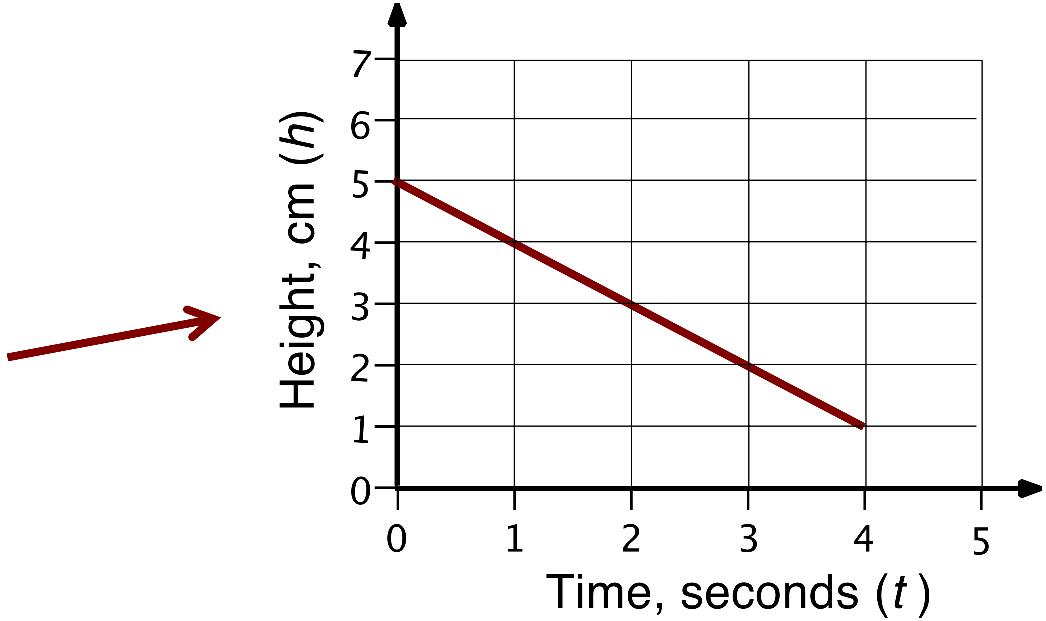
Liquid flowing out of the top prism 4



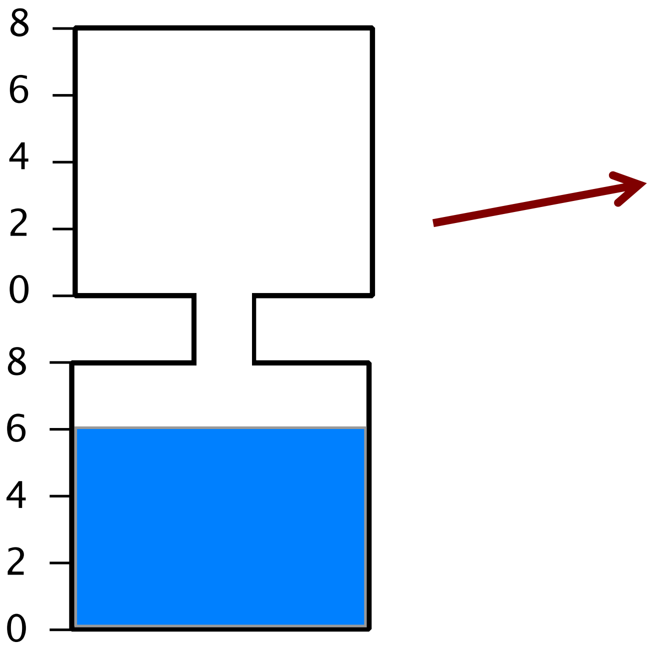


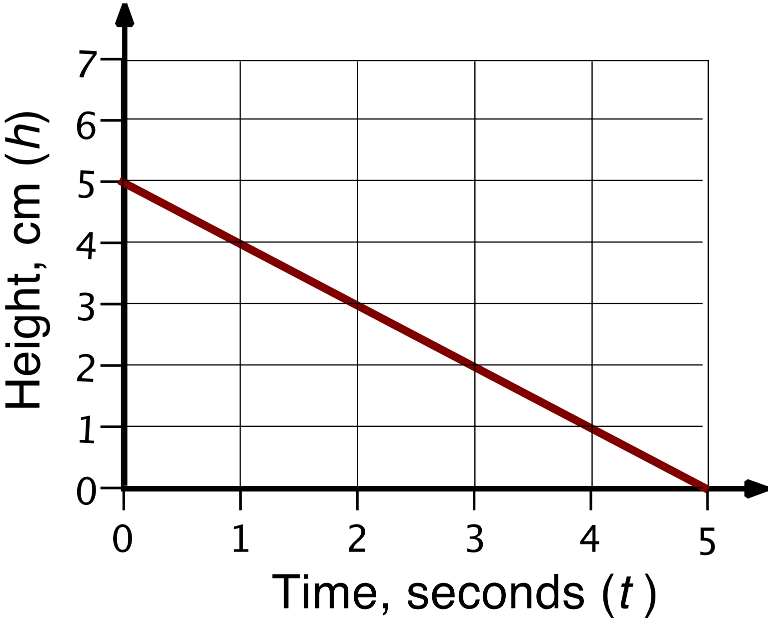
Liquid flowing out of the top prism 5



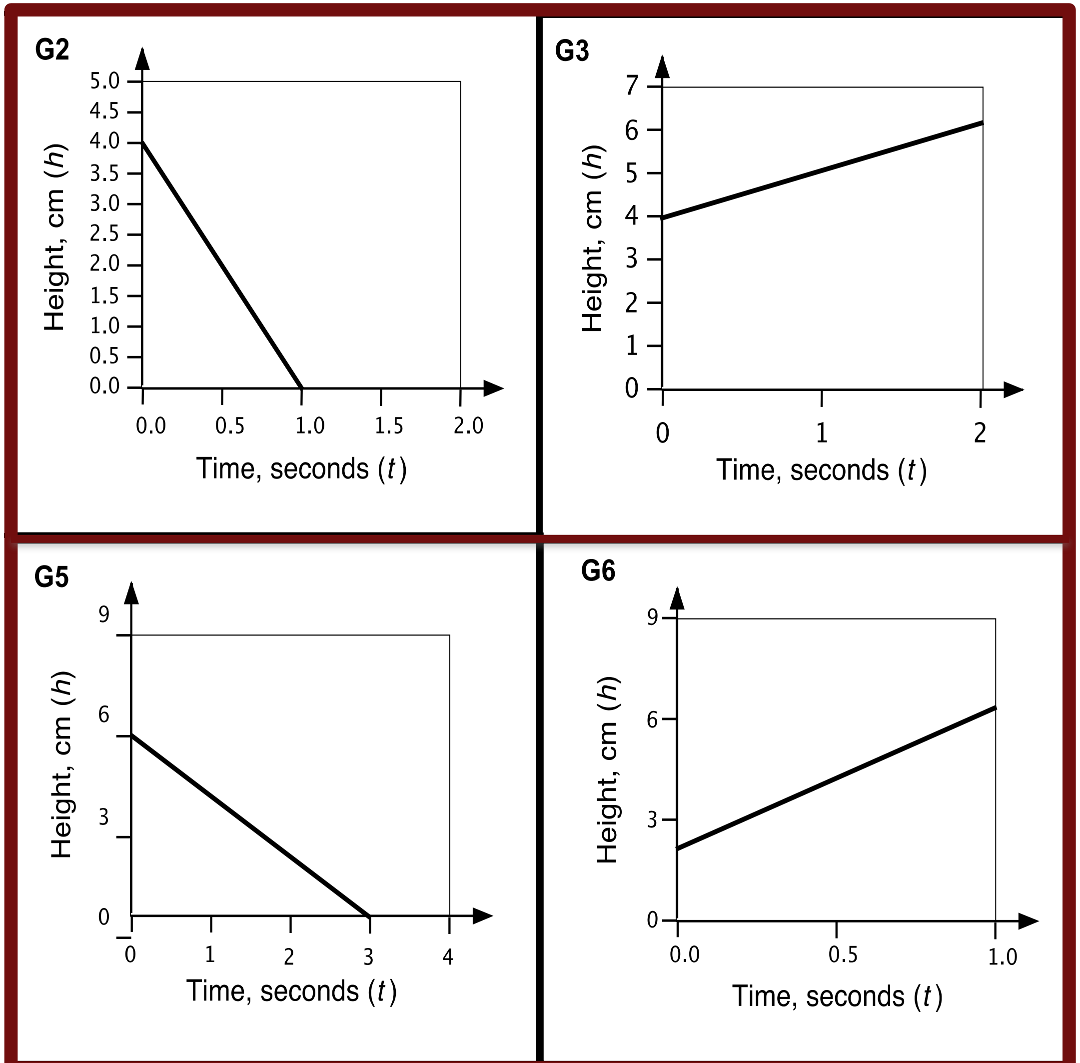


Liquid flowing out of the top prism 6





Graphs


G2
Height, cm (h)
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.0
0.5
1.0
1.5
2.0
Time, seconds (t)
G3
Height, cm (h)
7
6
5
4
3
2
1
0
0
1
2
Time, seconds (t)
G5
Height, cm (h)
9
6
3
0
0
1
2
3
4
Time, seconds (t)
G6
Height, cm (h)
9
6
3
0
0.0
0.5
1.0
Time, seconds (t)

Working Together

1. The graphs represent the flow of a liquid either out of the top prism or into the bottom prism of the container.

2. Take it in turns to match two cards that represent the movement of water in one container.

3. Place the cards next to each other, not on top, so that everyone can see.

4. When you match two cards, explain how you came to your decision.

5. Your partner should either explain that reasoning again in his or her own words, or challenge the reasons you gave.

6. Some graphs are missing information, such as a scale along an axis. You will need to add this scale.

You both need to be able to agree on and explain the match of every card.

Sharing Work

1. If you are staying at your desk, be ready to explain the reasons for your group's graph matches.

2. If you are visiting another group, copy your matches onto a piece of paper.

3. Go to another group's desk and check to see which matches are different from your own. If there are differences, ask for an explanation. If you still don't agree, explain your own thinking.

4. When you return to your own desk, you need to consider as a group whether to make any changes to your own work.

Mathematics Assessment Project

CLASSROOM CHALLENGES

This lesson was designed and developed by the
Shell Center Team
at the
University of Nottingham
Malcolm Swan, Clare Dawson, Sheila Evans,
Marie Joubert and Colin Foster
with
Hugh Burkhardt, Rita Crust, Andy Noyes, and Daniel Pead

It was refined on the basis of reports from teams of observers led by
David Foster, Mary Bouck, and Diane Schaefer
based on their observation of trials in US classrooms
along with comments from teachers and other users.

This project was conceived and directed for
MARS: Mathematics Assessment Resource Service
by
Alan Schoenfeld, Hugh Burkhardt, Daniel Pead, and Malcolm Swan
and based at the University of California, Berkeley

We are grateful to the many teachers, in the UK and the US, who trialed earlier versions
of these materials in their classrooms, to their students, and to
Judith Mills, Mathew Crosier, Nick Orchard and Alvaro Villanueva who contributed to the design.

This development would not have been possible without the support of
Bill & Melinda Gates Foundation
We are particularly grateful to
Carina Wong, Melissa Chabran, and Jamie McKee