CONCEPT DEVELOPMENT

Mathematics Assessment Project
CLASSROOM CHALLENGES
A Formative Assessment Lesson

# Representing and Combining Transformations

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley
Beta Version

# Representing and Combining Transformations

## MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to:

- Recognize and visualize transformations of 2D shapes.
- Translate, reflect and rotate shapes, and combine these transformations.

It also aims to encourage discussion on some common misconceptions about transformations.

## COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

G-CO: Experiment with transformation in the plane.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*:

1. Make sense of problems and persevere in solving them.
3. Construct viable arguments and critique the reasoning of others.
5. Use appropriate tools strategically.

## INTRODUCTION

The lesson unit is structured in the following way:

- Before the lesson, students work individually on an assessment task that is designed to reveal their current understandings and difficulties. You then review their work, and create questions for students to answer in order to improve their solutions.
- After a whole-class introduction, students work in small groups on a collaborative task.
- In a whole-class discussion, students review the main mathematical concepts of the lesson.
- Students return to their original task, and try to improve their own responses.

## MATERIALS REQUIRED

Each student will need two copies of the assessment task, *Transformations,* one copy of *L- Shapes* (printed on transparency film), and a map pin or thumbtack.

Each small group of students will need one copy each of *Card Set A: Shapes* and *Card Set B: Words* (cut up before the lesson), a copy of the transparency *Transformations* (printed on transparency film), a mini-whiteboard, a pen, an eraser, a glue stick and a large sheet of poster paper (optional).

You will need an overhead projector for demonstrating the transparencies to the whole class.

For an extension activity you will need several copies of *Card Set C: Additional Words*, and several pairs of scissors.

There are some projector resources to support discussion.

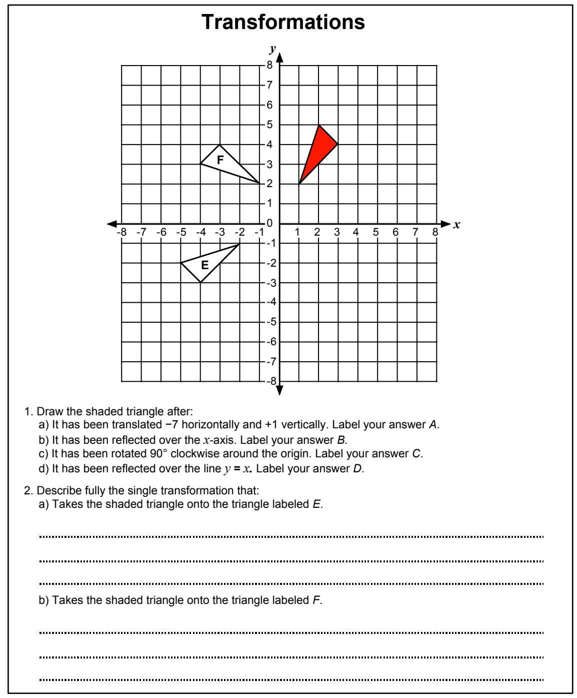
## TIME NEEDED

15 minutes before the lesson, a 1-hour lesson, and 10 minutes in the next lesson (or for homework). Timings are approximate and will depend on the needs of the class.

# BEFORE THE LESSON

## Assessment task: *Transformations* (15 minutes)

Set this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work, to find out the kinds of difficulties students have with it. You will then be able to target your help more effectively in the follow-up lesson.



**Transformations**



1. Draw the shaded triangle after:
   a) It has been translated −7 horizontally and +1 vertically. Label your answer *A*.
   b) It has been reflected over the $x$-axis. Label your answer *B*.
   c) It has been rotated 90° clockwise around the origin. Label your answer *C*.
   d) It has been reflected over the line $y = x$. Label your answer *D*.

2. Describe fully the single transformation that:
   a) Takes the shaded triangle onto the triangle labeled *E*.
   ......
   b) Takes the shaded triangle onto the triangle labeled *F*.
   ......

Give each student a copy of the assessment task *Transformations*.

> *Read through the questions and try to answer them as carefully as you can.*

It is important that, as far as possible, students are allowed to answer the questions without your assistance.

Students should not worry too much if they cannot understand or do everything, because in the next lesson they will engage in a similar task, which should help them. Explain to students that by the end of the next lesson, they should expect to be able to answer questions like these confidently. This is their goal.

## Assessing students' responses

Collect students' responses to the task, and make some notes on what their work reveals about their current levels of understanding. The purpose of doing this is to forewarn you of the difficulties students will experience during the lesson itself, so that you may prepare carefully.

We suggest that you do not score students' work. The research shows that this will be counterproductive as it will encourage students to compare their scores, and will distract their attention from the mathematics.

Instead, you can help students to make progress by asking questions that focus attention on aspects of their work. Some suggestions for these are given on the next page. These have been drawn from common difficulties in trials of this unit.

We suggest that you write your own list of questions, based on your own students' work, using the ideas that follow. You may choose to write questions on each student's work. If you do not have time to do this, you could write a few questions that will help the majority of students. These can then be displayed on the board at the end of the lesson.

| Common issues: | Suggested questions and prompts: |
|---|---|
| **Student confuses the terms 'horizontally' and 'vertically'**<br>For example: The student translates the shaded triangle −7 units vertically and +1 units horizontally in Q1a. | • Look at the start of the word 'horizontally'. What are we referring to when we talk about the horizon? Which way is this? |
| **Student translates rather than reflect the shape (Q1b)**<br>For example: The student has translated the shaded triangle vertically −7 units and so omitted to draw the mirror image. | • If you were to place a mirror on the x-axis, what would the reflected image look like? |
| **Student confuses the terms 'clockwise' and 'counterclockwise'**<br>For example: The student rotates the shaded triangle counterclockwise (Q1c.) | • Think about the direction of the hands on a clock. This direction is 'clockwise'. |
| **Student ignores the center of rotation and rotates from a corner of the shaded triangle**<br>For example: The student rotates the shaded triangle around the point (1, 2) (Q1d.) | • Where is the center of rotation?<br>• Mark the center of rotation and draw a line to a corner of the shape. Where will this line be once it has been rotated? |
| **Student uses an inefficient combination of transformations**<br>For example: The student describes the transformation in Q2a as "a reflection over the *y*-axis, followed by a rotation 90° counterclockwise around (−1, 2), followed by a translation −1 unit horizontally and −3 unit vertically". | • Is there a single transformation that will take the shaded triangle directly to the triangle labeled E? |
| **Student correctly answers all the questions**<br>The student needs an extension task. | • Find a combination of two transformations that could be replaced by a single one. |

## SUGGESTED LESSON OUTLINE

### Whole-class interactive introduction (15 minutes)

Give each student the transparency *L-Shapes,* and a pin (to help find centers of rotation).

Using transparencies encourages students to test different transformations. Working dynamically should deepen students' understanding of transformations in a way that simply drawing shapes on a graph does not.

Introduce the lesson by using slides P-1, P-2, and P-3 of the projector resource.

Ask the students where they think the image of the *L-Shape* will be after it has been translated, reflected, or rotated in different ways:

*Where will the L-Shape be if it is translated −2 units horizontally and +1 units vertically?*

*Where will the L-Shape be if it is reflected over the line $x = 2$?*

*Where will the L-Shape be if it is rotated through 180° around the origin?*

Ask volunteers to demonstrate their answers by placing their grid and *L-Shape* on the overhead projector. Discuss these positions with the rest of the class, and encourage students to challenge their peers if they think the *L-Shape* has been positioned incorrectly.

Once the correct position has been agreed upon, move on to the next transformation.

You may also want to move the *L-shape* to a different position on the grid, and ask students:

*What transformation will move the L-shape to this new position? Show me.*

### Collaborative work (30 minutes)

Ask students to work in groups of two or three.

Give each group *Card Set A: Shapes* and *Card Set B: Words* and a copy of the transparency *Transformations*.

Introduce the activity:

*You are now going to continue to transform L-shapes.*

*You've got six shape cards, each showing a different L-shape, and eight word cards each of which describes a different transformation.*

*Take turns to link two shape cards with a word card. Make sure the arrow goes in the right direction! Each time you do this explain your thinking clearly and carefully.*

*Your partner should then either explain their reasoning again in his or her own words, or challenge the reasons you gave.*

*It is important that everyone in the group understands the placing of a word card between two shape cards.*

*Ultimately, you want to make as many links as possible. Use all the shape cards, and all the word cards if possible.*

You may wish to use Slide P-4 of the projector resource to display these instructions.

You have two tasks during the paired work: to make a note of student approaches to the task, and to support student reasoning.

### Make a note of student approaches to the task

Listen and watch students carefully. In particular, listen to see whether students are addressing the difficulties they experienced in the assessment. For example, are students having difficulty rotating a shape around (2,0) or reflecting a shape over the lines $y = x$ and $y = -x$? You can use information about particular difficulties as a focus for the whole-class discussion later in the lesson.

### Support student reasoning

Use the questions in the *Common issues* table to help address misconceptions.

Encourage students to explain carefully how they have made each connection.

> *Lian, please explain why you've linked these two shapes with this transformation.*
>
> *Laura, can you repeat Lian's explanation in your own words?*

Ask students:

> *How does folding the L-Shape along the line of reflection help when reflecting the shape?*
>
> *How does drawing a line from the center of rotation to a corner of the shape help when rotating the shape?*

Students who are struggling should be encouraged to concentrate on linking Shape Cards A, B, C and D.

### Further transformations

Once students have completed their arrangement of cards, give them a copy of *Card Set C: Additional Words*, and a pair of scissors.

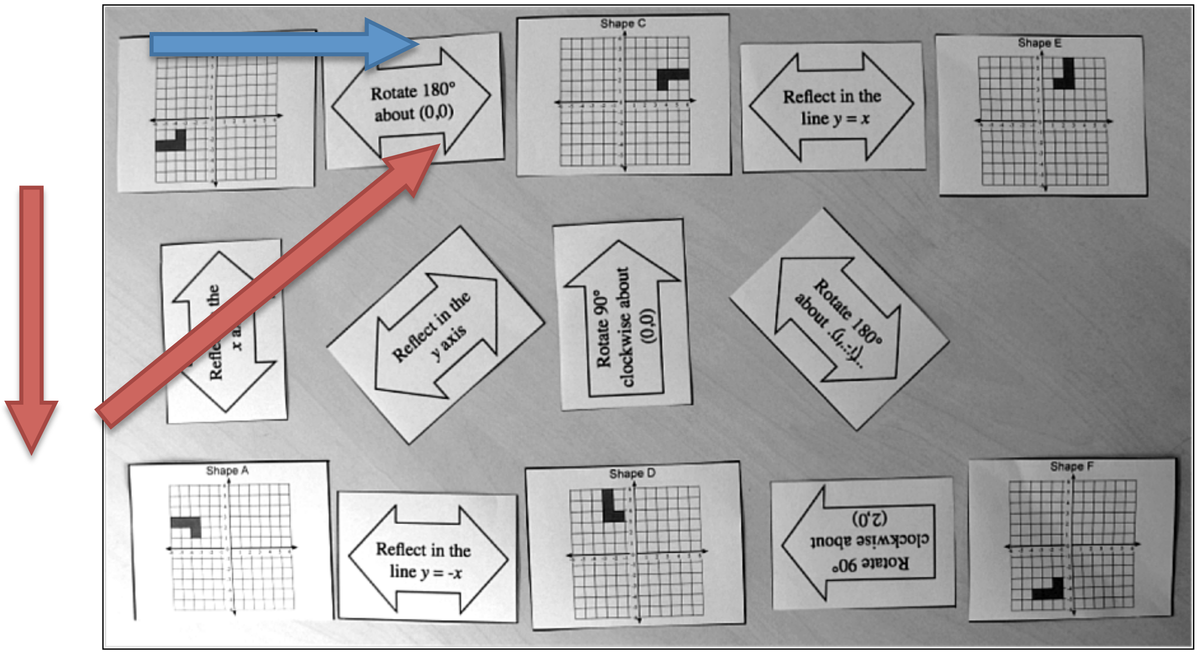
Ask students to add an appropriate transformation, where possible, between any shape cards that has not yet been connected.

On completion, students may then glue the cards on a poster. They will need a glue stick and a sheet of large poster paper to do this.

## Extension task

If a group of students successfully completes the task:

> *Can you find a combination of two transformations that could be replaced by a single one?*
>
> [*For example, reflect B over the $x$-axis B onto A, then reflect A over the $y$-axis onto C. These two transformations can be replaced by a single transformation: rotate B through180° around the origin onto C. This can be seen on the example arrangement below.*]




Students should be encouraged to investigate whether or not this is always the case:

*For any shape, will this combination of transformations always replace this single one?*

A proof would involve considering what would happen to the general point $(x, y)$. Under a reflection over the $x$-axis, this would go to $(x, -y)$. After a further reflection over the $y$-axis, this would become $(-x, -y)$. This is the same as the general point $(x, y)$ being rotated through180° around the origin.

Students should be encouraged to look for other possible combinations in their card arrangements in the same way.

## Whole-class discussion (15 minutes)

Give each group of students either a mini-whiteboard, pen, and eraser, or a piece of squared paper.

Use Slides P-5 and P-6 of the projector resource to support a whole-class discussion.

Ask students to do the following transformations using the coordinate grid on the transparency *Transformations*, then to write the new coordinate on their mini-whiteboard:

*Use the transparency Transformations. Mark the coordinate (1, 4) on the coordinate grid.*

*Show me the new coordinates of the point (1, 4) after it is:*

- *Reflected over the x-axis.* — *(1, −4)*
- *Reflected over the y-axis.* — *(−1, 4)*
- *Rotated through 180° around the origin.* — *(−1, −4)*
- *Reflected over the line $y = x$.* — *(4, 1)*
- *Reflected over the line $y = -x$.* — *(−4, −1)*
- *Rotated through 90° clockwise around the origin.* — *(4, −1)*
- *Rotated through 90° counterclockwise around the origin.* — *(−4, 1)*

You may like to repeat this with a general starting point $(x, y)$.

*Show me the new coordinates of the general point $(x, y)$ after it is:*

- *Reflected over the $x$-axis.* — $(x, -y)$
- *Reflected over the $y$-axis.* — $(-x, y)$
- *Rotated through 180° around the origin.* — $(-x, -y)$
- *Reflected over the line $y = x$.* — $(y, x)$
- *Reflected over the line $y = -x$.* — $(-y, -x)$
- *Rotated through 90° clockwise around the origin.* — $(y, -x)$
- *Rotated through 90° counterclockwise around the origin.* — $(-y, x)$

It may be helpful to write the new coordinates on the board, to be able to extend discussions to include combinations of transformations:

*What is the single transformation that will produce the same result as:*

- *A rotation of 90° clockwise around the origin, followed by a reflection in the y-axis?*

*[This is a reflection in the line $y = -x$.]*

*Show me two transformations that can be written as a single direction.*

*Show me two transformations that cannot be written as a single direction. Can you change the starting point of the shape so that it can be written as a single direction?*

## Improving individual solutions to the assessment task (10 minutes)

Return their original assessment task *Transformations* to the students, together with a blank copy of the task.

*Look at your original responses and think about what you have learned this lesson.*

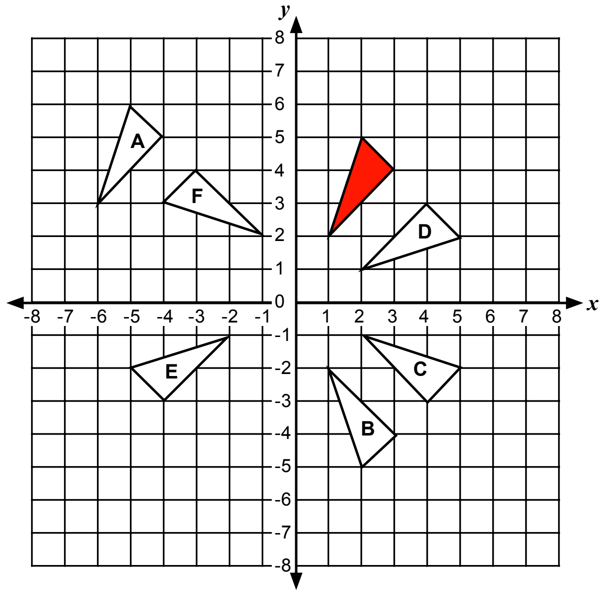
*Using what you have learned, try to improve your work.*

If you have not added questions to individual pieces of work then write your list of questions on the board. Students should select from this list only those questions they think are appropriate to their own work.

If you find you are running out of time, you could set this task in the next lesson or for homework.

## SOLUTIONS

### Assessment Task: *Transformations*




2a. Reflection over the line $y = -x$.

2b. Rotation 90° counterclockwise around (0, 1).

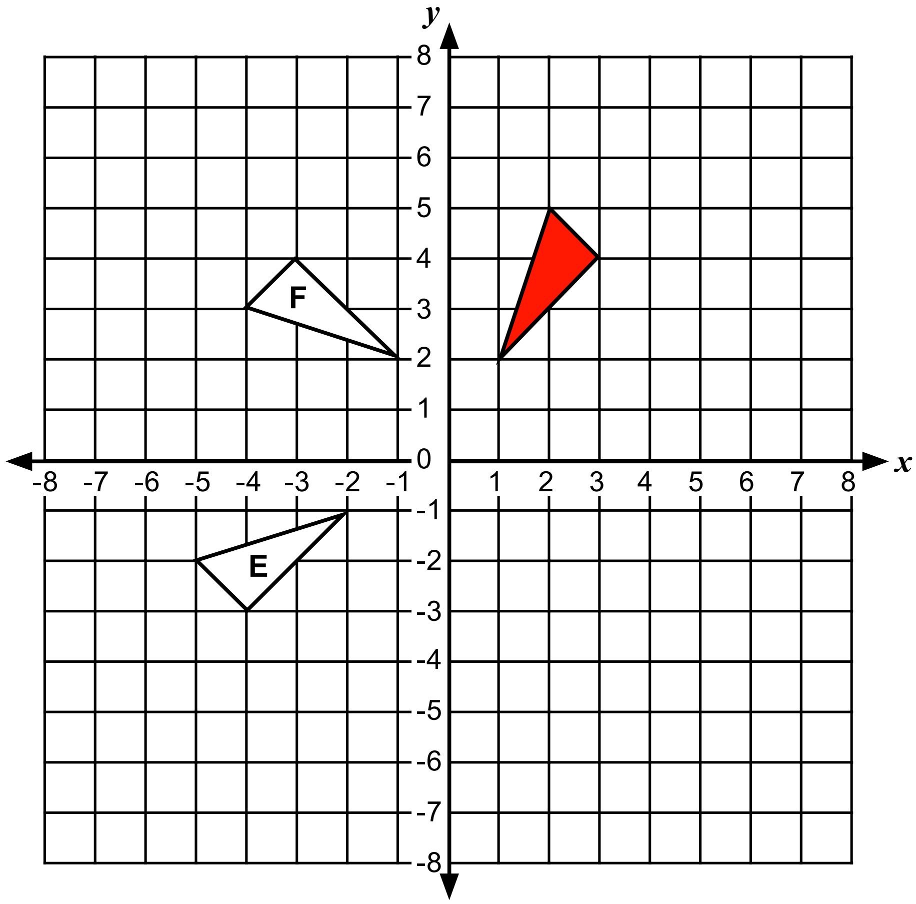
3. Reflection over the line $y = x$.

#### Collaborative work

The following connections exist between pairs of shape cards:

| Pairs of shapes | Transformation | Pairs of shapes | Transformation |
|---|---|---|---|
| A onto B | Reflection over the x-axis. | B onto F | Translation +2 units horizontally and –2 units vertically. |
| F onto D | Clockwise rotation of 90° around (2,0). | D onto E | Reflection over the y-axis. |
| E onto C | Reflection over the line y = x. | C onto A | Reflection over the y-axis. |
| A onto D | Reflection over the line y = –x. | C onto B | Rotation of 180° around the origin. |
| A onto E | Clockwise rotation of 90° around the origin. | B onto D | Clockwise rotation of 90° around the origin. |
| D onto C | Clockwise rotation of 90° around the origin. | B onto E | Reflection over the line y = –x. |

# Transformations



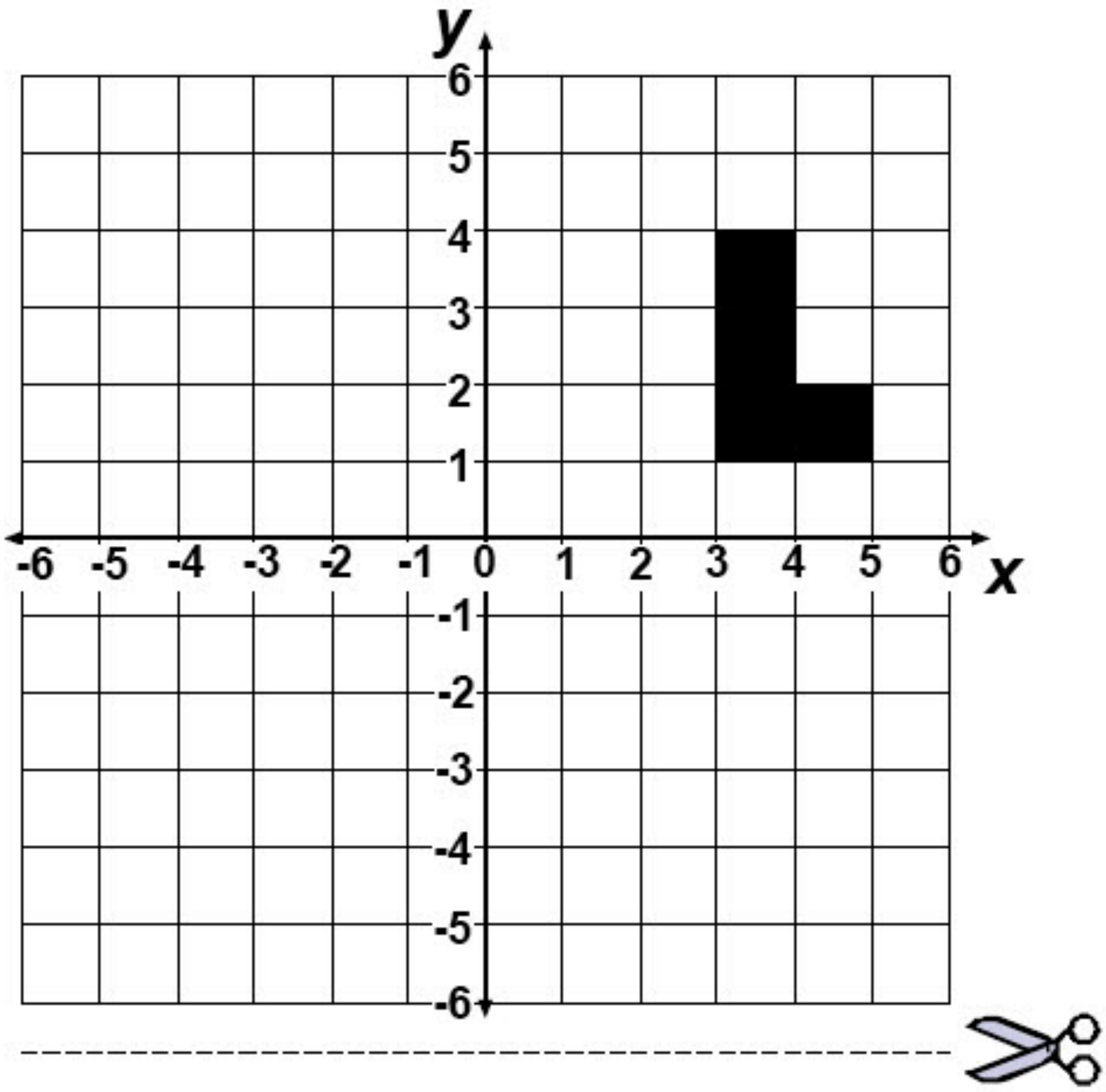

1. Draw the shaded triangle after:
   a) It has been translated −7 horizontally and +1 vertically. Label your answer *A*.
   b) It has been reflected over the $x$-axis. Label your answer *B*.
   c) It has been rotated 90° clockwise around the origin. Label your answer *C*.
   d) It has been reflected over the line $y = x$. Label your answer *D*.

2. Describe fully the single transformation that:
   a) Takes the shaded triangle onto the triangle labeled *E*.

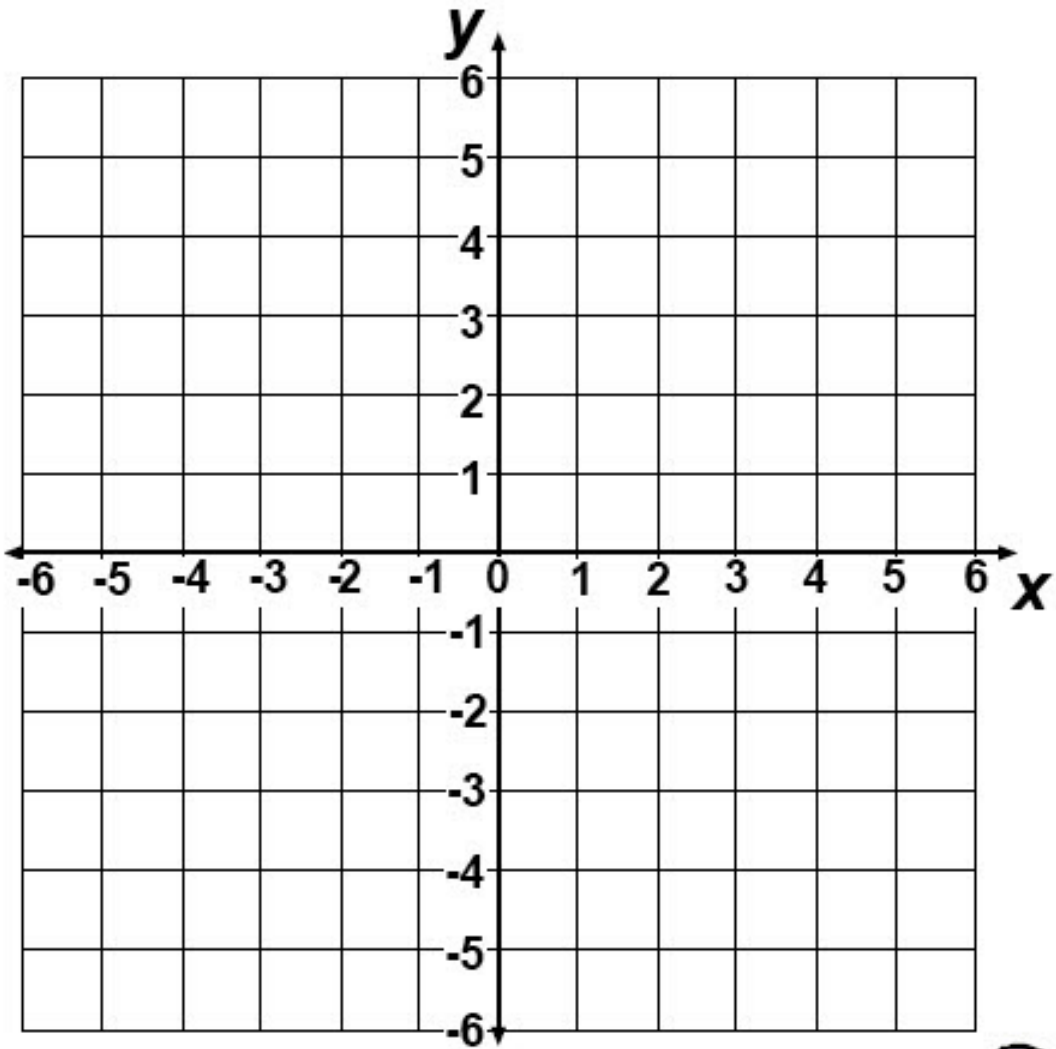
   b) Takes the shaded triangle onto the triangle labeled *F*.

3. Describe a single transformation that has the same effect as rotating a shape 90° clockwise around the origin, then reflecting the result over the $x$-axis.

# Transparency: L-Shapes

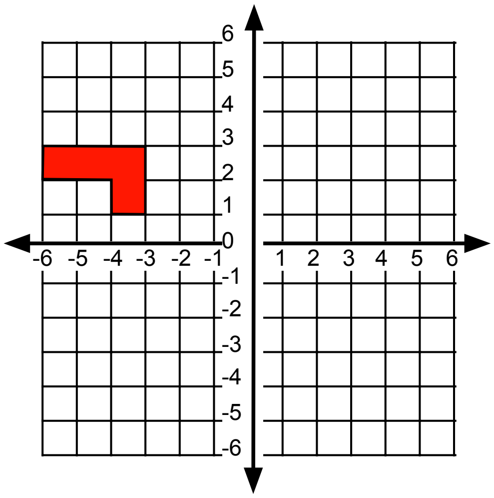



# Transparency: Transformations

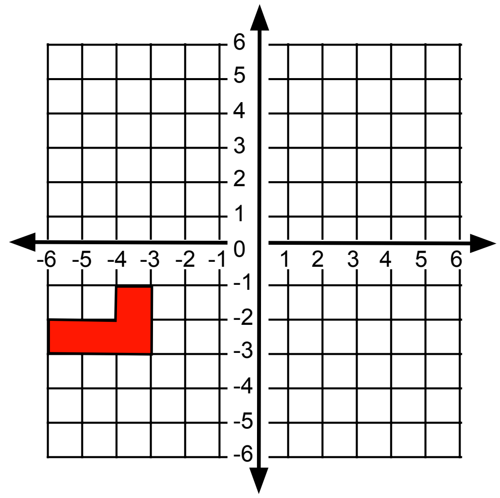



# Card Set A: Shapes

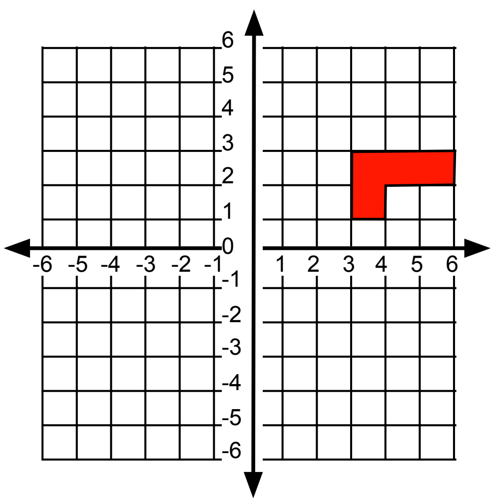
Shape A



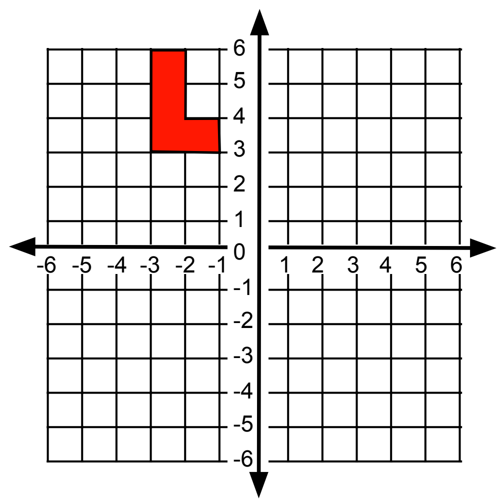

Shape B



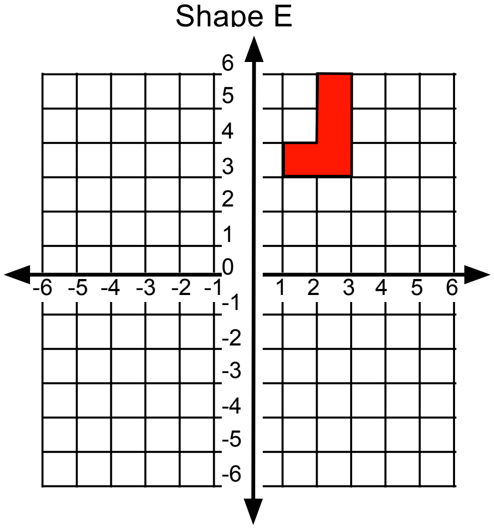

Shape C



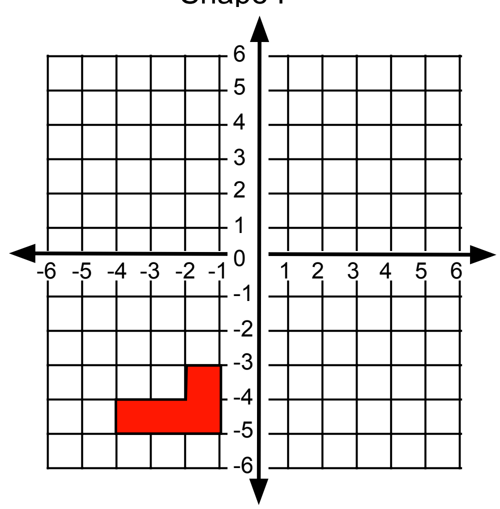

Shape D




Shape E




Shape F



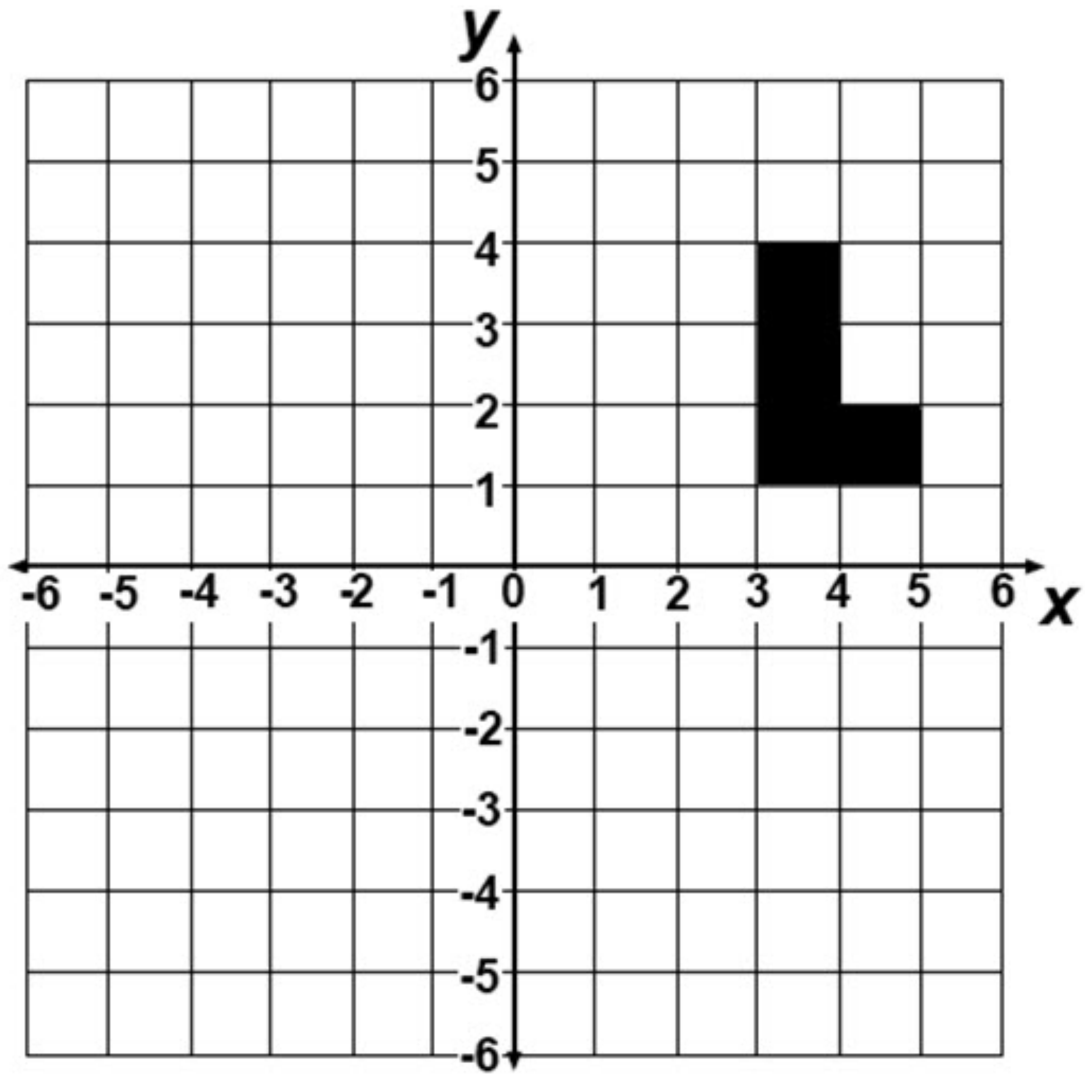
# Card Set B: Words

| | |
|---|---|
| Reflect over the $x$-axis | Reflect over the $y$-axis |
| Reflect over the line $y = x$ | Reflect over the line $y = -x$ |
| Rotate 180° around the origin | Rotate 90° clockwise around the origin |
| Rotate 90° clockwise around (2, 0) | Translate +2 horizontally −2 vertically |

# Card Set C: Additional Words

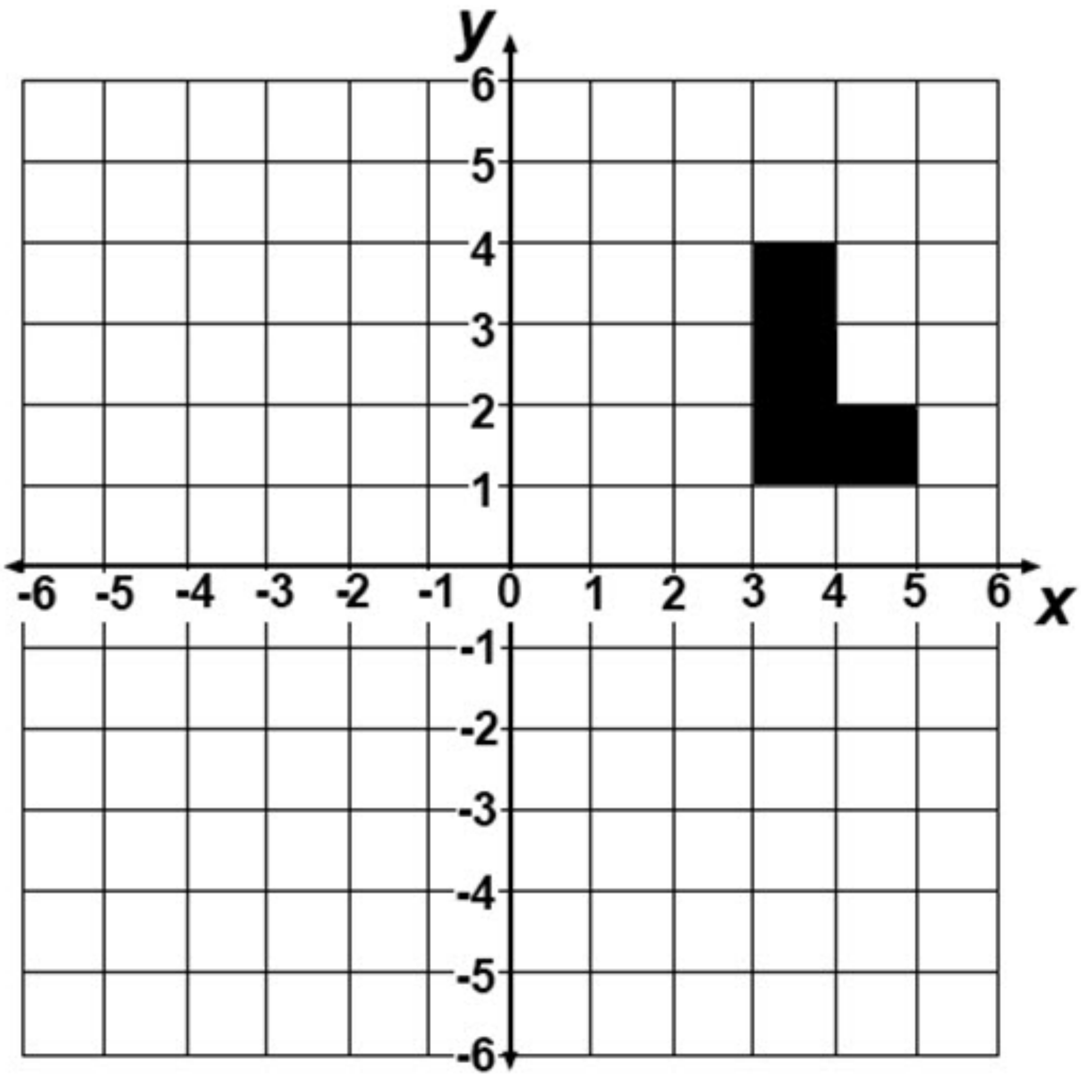
| | |
|---|---|
| Reflect over the ................ | Reflect over the ................ |
| Reflect over the ................ | Reflect over the ................ |
| Rotate 180° around ............ | Rotate 90° clockwise around ................ |
| Rotate 90° counterclockwise around ................ | Translate ................ ................ |

# Translation



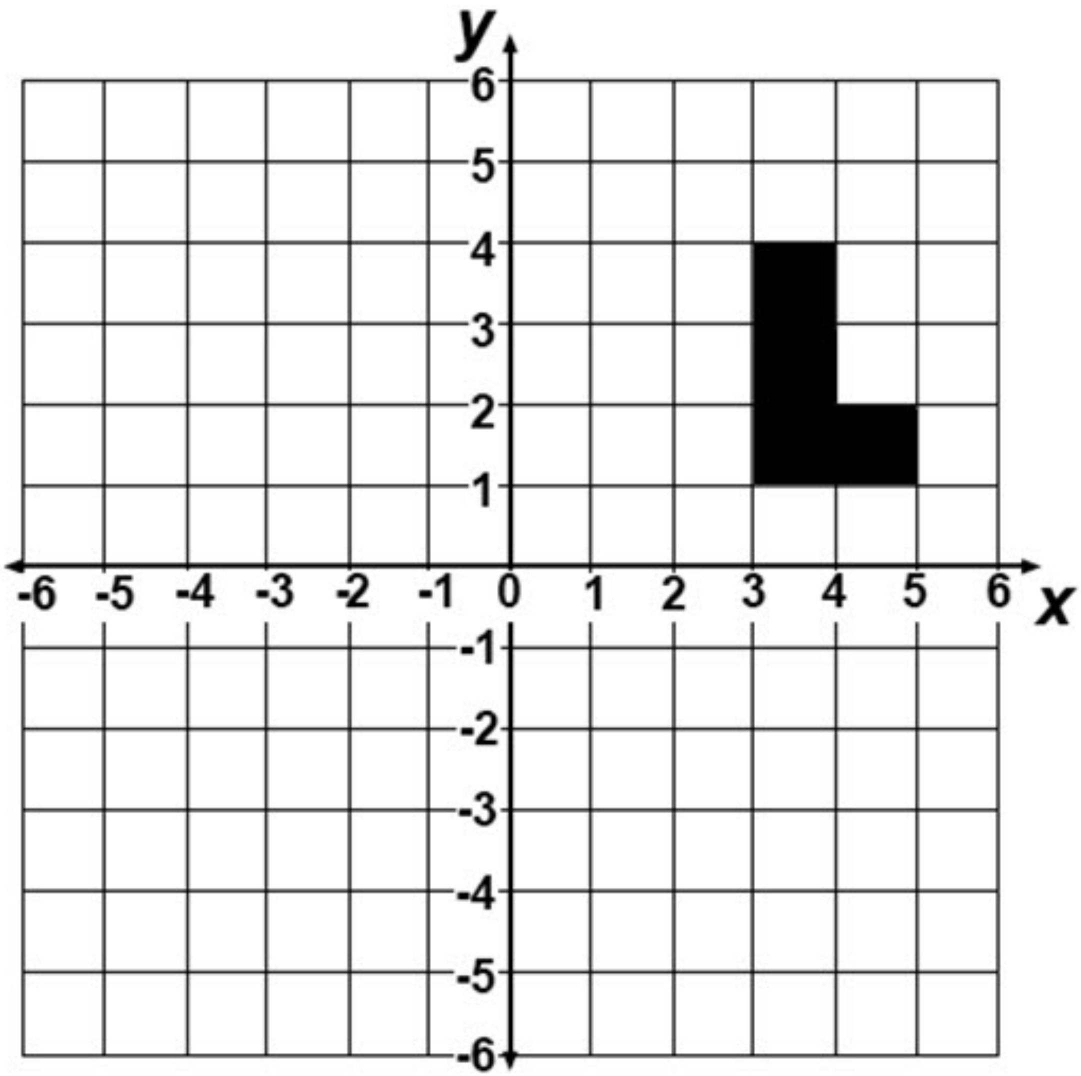

Where will the L-shape be if it is translated by −2 horizontally and +1 vertically?

# Reflection




Where will the L-shape be if it is reflected over the line $x = 2$?

# Rotation




Where will the L-shape be if it is rotated through 180°around the origin?

# Matching Cards

- Take turns to match two shape cards with a word card. Each time you do this, explain your thinking clearly and carefully.
- Your partner should then either explain that reasoning again in his or her own words, or challenge the reasons you gave.
- It is important that everyone in the group understands the placing of a word card between two shape cards .
- Ultimately, you want to make as many links as possible.
- Use all the shape card, and all the word the cards if possible.

# Starting point (1, 4)

Show me the new coordinates of the point (1, 4) after it is:

- Reflected over the $x$-axis
- Reflected over the $y$-axis
- Rotated through 180°about the origin.
- Reflected over the line $y = x$.
- Reflected over the line $y = -x$.
- Rotated through 90°clockwise about the origin.
- Rotated through 90°counterclockwise about the origin.

# General starting point ($x$, $y$)

Show me the new coordinates of the point ($x$, $y$) after it is:

- Reflected over the $x$-axis
- Reflected over the $y$-axis
- Rotated through 180°about the origin.
- Reflected over the line $y = x$.
- Reflected over the line $y = -x$.
- Rotated through 90°clockwise about the origin.
- Rotated through 90°counterclockwise about the origin.

Mathematics Assessment Project

# CLASSROOM CHALLENGES

This lesson was designed and developed by the
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It was refined on the basis of reports from teams of observers led by
**David Foster, Mary Bouck, and Diane Schaefer**
based on their observation of trials in US classrooms
along with comments from teachers and other users.

This project was conceived and directed for
MARS: Mathematics Assessment Resource Service
by
**Alan Schoenfeld, Hugh Burkhardt, Daniel Pead, and Malcolm Swan**
and based at the University of California, Berkeley

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