The Value of Debts

Promote discussions on integers in real life by using positive and negative numbers in the context of net worth.

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Research has shown that students should be given the opportunity to explore mathematical concepts by building on their knowledge and focusing on mathematical reasoning (Goldsmith and Schifter 1997). When students represent ideas, make conjectures, collaborate with others, and give explanations and arguments, they are using mathematical reasoning (NCTM 2000). Wheatley (1991) goes on to say that when students give explanations and make conjectures, they elaborate, clarify, and reorganize their thinking. During discussions with others, students are learning to understand the relationship among ideas that in turn give meaning to mathematical procedures; these ideas then support conceptual understanding (Eisenhart et al. 1993).

Certain teaching practices can support students' mathematical reasoning. The topic highlighted in this article—integers—was selected because it poses unique challenges. When students are reasoning with negative numbers for the first time, they are encountering quantities that cannot be physically realized. Historically, the topic of integers, especially negative numbers, has been a challenging concept for mathematicians (Hefendehl-Hebeker 1991).

A person's financial net worth provided the context for these


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How to Foster **Mathematical Reasoning**

Give encouragement in the form of explanations, conjectures, and both efficient and sophisticated solutions to promote student understanding.

ENCOURAGE STUDENTS TO GIVE CONCEPTUAL EXPLANATIONS

- Help students make sense of key mathematical concepts by making the meaning of the concepts explicit.
- Use appropriate representations to support reasoning and help students communicate conceptual explanations.

ENCOURAGE EFFICIENT SOLUTIONS

- Design tasks so that students are likely to find an efficient solution.
- Challenge students to think of a more efficient method.
- Discuss the idea underlying the efficiency.

ENCOURAGE STUDENTS TO MAKE CONJECTURES AND PROVE THEM

- Pose questions that give students opportunities to generalize their ideas.
- Encourage students to generalize the ideas that are presented in the tasks.
- Ask students to label their conjectures and prove them.

ENCOURAGE DIFFERENT SOLUTIONS

- Design tasks that have multiple correct answers.
- Ask students to analyze different answers.
- Guide students through different solutions, which may or may not have come from classmates.

ENCOURAGE SOPHISTICATED SOLUTIONS

- Ask questions to help students understand the steps in a sophisticated solution.
- Give all students opportunities to understand a sophisticated method by restating students' words and using different representations.
- Allow students to be part of the solution process by analyzing a sophisticated solution together as a class.

activities. The focus was on helping students understand how assets and debts were quantitatively composed in a net worth statement and how integer operations worked in this context (Stephan 2009). These integer operations were primarily addition and subtraction, but multiplication was also introduced. The instruction followed the theory of realistic mathematics education (RME), which states that an activity should focus on an experientially real context (Gravemeijer 1994). *Experientially real* means that students do not necessarily have to experience the activity themselves but should be able to imagine acting within the context. This experience allowed students to use their informal knowledge of mathematics as a starting point in developing progressively more formal mathematical reasoning.

This study evolved out of dissertation research that sought to extract the planning and classroom practices of an expert mathematics teacher who followed the tenets of NCTM's Standards (Akyuz 2010). The five-week study was conducted during a unit on integers in a seventh-grade classroom in Central Florida. Of the twenty students in the class, thirteen were boys and seven were girls. Three of the students had been identified as having mild learning disabilities.

The data set included interviews with the teacher, audiotaped and videotaped classroom sessions, field notes, teacher notes, research meetings, and a collection of student work. The teacher's frequently repeating actions were first identified and then compared and contrasted with other occurrences in the data set. The consistent actions reflected the teacher's practice. Once these actions were extracted, they were grouped under different practices, which were primarily generated from the teacher

Fig. 1 A vertical number line, showing the positive area in black and the negative area in red, provided a context for the conceptual explanation.

Paris's net worth is in the red: −$20,000.
Nicole's net worth is in the red: −$22,000.

1. Who is worth more?
2. How much more?
3. Show this on the net worth line.

Fig. 2 Given James's net worth statement and some transactions, students could compare solution paths and consider which were more efficient.

James's Worth Statement

1. Asset 1: +$200
2. Asset 2: +$145
3. Debt 1: −$200
4. Debt 2: −$650
5. Asset 3: +$650
6. Asset 4: +700
7. Debt 3: −$50

notes and the literature. The most important practices that supported student reasoning involved encouraging students to give conceptual explanations, to think of efficient solutions, to make conjectures and prove them, and to provide different and sophisticated solutions. (See the **sidebar** at left.)

These practices are not novel, and most readers may already be familiar with them. This article illustrates how the expert teacher effectively incorporated these practices into her teaching. Although the instruction involved assets and debts, the practices can be generalized to other topics. These practices did not occur in a linear order during the instruction but were interspersed throughout. The following sections describe these practices and provide classroom examples.

ENCOURAGING STUDENTS TO GIVE CONCEPTUAL EXPLANATIONS

Students' ability to give *conceptual explanations*—to demonstrate their understanding of a subject by discussing it—allows the teacher to evaluate whether students truly understand the concept or whether their understanding is superficial. The expert teacher frequently encouraged students to give conceptual explanations. For instance, once the student became comfortable with the meaning of assets and debts, the teacher asked, "Paris's net worth is –$20,000, and Nicole's net worth is –$22,000. Find the gap between them." During the discussion, some students had difficulty ordering the negative numbers on a vertical number line. To help students, the teacher encouraged them to give conceptual explanations:

Teacher: Charlie, you had an idea that –$22,000 is below –$20,000. Why is this number below the other? Can you say it again? Perhaps that might help Nathan, too.

Charlie: Because –$20,000 is closer to being out of debt than –$22,000.

Teacher: Did you hear that? –$20,000 is closer to being out of debt than –$22,000. Can anybody state it differently?

Brad: The reason –$22,000 is below because she is further down the hole.

Teacher: Do you know what he is talking about? What do you mean by *further down in the hole*?

Brad: Like you owe more money than the other person.

Teacher: So are you saying that the further down you are going in the hole [the teacher gestures downward], the worse your debts get? What do you think about this idea?

The teacher posed this question using a vertical number line, on which the positive and negative areas were shown in black and red (see **fig. 1**). The students could connect this tool with problems that they solved earlier by associating *being in the black* with having assets and *being in the red* with having debts. Thus, this vertical number line served as a tool to support students' imagery.

During the discussion, Charlie's first explanation produced a satisfactory answer, and the teacher sought another explanation to ensure that students understood the ordering of the numbers. Brad's use of the term "further down the hole" resulted in the teacher asking for clarification of the term. Brad said it meant owing more, having more debt. The teacher then summarized Brad's main points and asked the class what they thought about that idea.

This discussion illustrated how the teacher repeatedly encouraged students to give conceptual explanations and clarify statements. This teacher behavior was prevalent throughout the entire sequence. She also encouraged the use of appropriate representations (e.g., a number line) to support reasoning and help students with their conceptual explanations.

ENCOURAGING EFFICIENT SOLUTIONS

One important role of a teacher is to encourage students to seek efficient solutions. These solutions not only save time and effort but also help students avoid errors that may occur during long and unnecessary computations. In one activity, students were asked to find a person's net worth and then apply a given transaction to this net worth (see **fig. 2**). The excerpt below occurred during a class discussion:

Danny: I just added all the positives and got $1695; that is total assets [teacher writes on the board]. I got $900 for total debts. And then I got $795 [the teacher writes 1695 – 900 = 795 on the board].

Teacher: Anybody have a quicker way? If you have to add, how many things do you have to add together? [The teacher shows the positive numbers in **fig. 2**.]

Tisha: It is not harder [suggesting that this slower method is not hard].

Teacher: Okay, if you think that is quick, it is fine. Does anyone have a way that is quicker than that?

Brad: Here is $200 and debt of $200; cross that off. And then $650 and –$650, cross that off, too. [The teacher crosses through those numbers on the board.] And then I added $700 plus $145 and got $845. [The teacher combines those two numbers with arrows and writes $845.]

Teacher: Where did you go next?

Brad: Subtract $50 and got $795. [The teacher writes 845 – 50 = 795.]

Teacher: Which one is quicker to you?

Students: I like Brad's way.

In solving this problem, most students continued to use their original strategy of totaling the assets and totaling the debts and subtracting assets from debts, as Danny described. However, the teacher's primary goal was for students to explore the idea of additive inverse, a more efficient solution. When she prompted the class for a quicker way, Brad explained his method and claimed that his method was quicker because he was able to cross out the amounts that were the same but had opposite signs and add fewer numbers. To clarify, the teacher crossed through the inverses as Brad described his solution and asked which solution was quicker. Most agreed that Brad's solution was quicker and better.

The teacher needs to design tasks with multiple solutions and encourage students to compare solution paths and consider their efficiency.

Students have a tendency to use familiar but inefficient solution strategies. Therefore, the teacher needs to design tasks with multiple solutions and encourage students to compare solution paths and consider their efficiency. The ideas underlying efficiency should then be discussed. As this scenario shows, the selection of the task plays a vital role in the application of this practice.

ENCOURAGING STUDENTS TO MAKE AND PROVE CONJECTURES

Making conjectures allows students to strengthen their mathematical reasoning. To encourage conjecturing, a teacher can pose questions or assign tasks that offer students opportunities to generalize their ideas. The expert teacher asked students to determine whether verbal descriptions of given transactions were good or bad, such as "Christian took away an asset of $50 from his net worth statement" and "Donnie takes away a debt of $500." To make it easier for students to notice the patterns, the teacher introduced some symbols, such as the minus sign for taking away (as an operation) and a negative sign for the debt itself (as a quantity). Stuart noticed a pattern and shared his conjecture:

Stuart: I think we should make a class conjecture about this. If we have the signs both inside and outside the parentheses that are the same, then you are making money—if they are like negative sign and negative sign. You are making money because you are taking away debt, and it is the same if you have both positive signs inside and outside. [The teacher writes on the board: "–(–), +(+); you are making money, good."]

Teacher: Do you want to finish, Stuart? Is it the only part of your conjecture?

Stuart: If you have positive outside and negative inside, it is not good, it is bad. If you have a negative outside and positive inside, it is bad, too. [The teacher writes: "+(–), –(+), bad."]

Stuart first made the conjecture that –(–) and +(+) are good, because the person makes money in both cases. With encouragement from the teacher, Stuart added that if the signs are different, as in –(+) and +(–), then it is bad because the person is either taking away an asset or adding a debt.

To further the discussion, the board was divided into two sections: "conjectures" and "theories." When a student, such as Stuart, made a proposition, it was posted under "conjecture" and listed with the student's name. Giving students a sense of ownership can motivate them to make and share their conjectures. Once the proposition was proved, it was moved to the "theory" section. The physical separation of conjectures and theories can be an impetus for students to prove their conjectures.

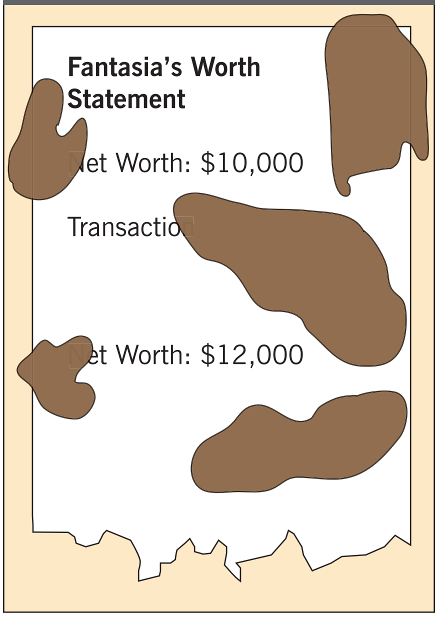
ENCOURAGING DIFFERENT SOLUTIONS

Encouraging students to explore different solutions may help them improve their skills to develop their own solution strategies rather than rely on a single common solution. This exploration emphasizes the importance of devising one's own solution instead of memorizing someone else's solution. For example, in one activity students were asked to write the same transactions in different ways. The teacher emphasized efficient and alternative solutions.

The excerpt below involved the possible answers for Fantasia's missing transaction, which was covered by a coffee spill (see **fig. 3**). In this question, the original and the final net worths were $10,000 and $12,000, respectively, and the transaction was missing. As the discussion continued, students found simple answers, such as +$2000, as well as more complicated transactions, such as –(+$3000) + (+$5000). For each unique answer, the teacher asked the students to analyze the transaction and state whether they agreed or not. The students used assets and debts; the number line; and previous conjectures, such as that –(–) and +(+) are good decisions. The teacher introduced an alternative solution that was not suggested by any classroom member:

Teacher: Let me share with you what one student in our last class period did. Started off with $10,000 and added plus $500 plus another $500. . . . [She writes $10,000 + (+$500) + (+$500) + (+$500) + (+$500) on the board.] Is that possible?

Fig. 3 The Coffee Spill problem allowed different solution paths to determine the missing transaction.



Students: Yes.

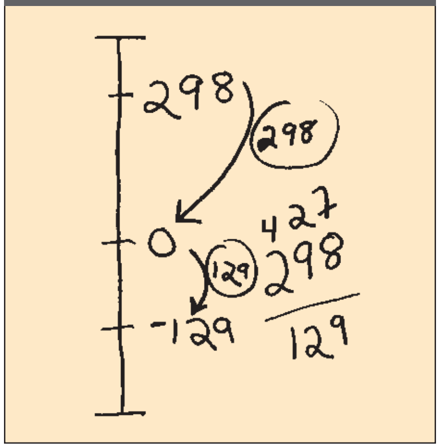
Teacher: It equals $12,000. It is doable, isn't it? He just thought about breaking $2000 into $500s. [She writes $10,000 + 4(+$500) on the board.] What do you think that means?

Students: Four $500s.

Teacher: Four $500 assets are being added. Do you see that? This is going to be a shortcut; $10,000 net worth, and you add four $500 assets, and then you will get $12,000 at the end. If you ever see something like that with the multiplication, you'll know that is how many sets of assets or debts that you are doing in the transaction.

First, the teacher gave students an opportunity to think of different solutions; next, she introduced multiplication as an alternative to their solutions and supported their understanding of it. This helped students better connect the new concept with familiar concepts. To motivate students to investigate different solutions, the teacher can design tasks that have multiple

Fig. 4 Mark's sophisticated solution, which involved first consuming the positive numbers and then the negative numbers, required an explanation on the board.



correct answers and guide students through different solutions, which may not have been discussed by students.

ENCOURAGING SOPHISTICATED SOLUTIONS

Sophisticated solutions can be defined as those that require a higher level of thinking than ordinary solutions; they can give students a new perspective. Once students solve a problem in a particular way, they may routinely carry out operations without thinking about their meaning. This occurred when students continued to add and subtract numbers without using the sophisticated solution of canceling out the additive inverses, which was also more efficient. A sophisticated solution does not necessarily have to be more efficient, but it should give a deeper insight into the nature of the problem.

The excerpt below was taken from a discussion about finding a new net worth, given an original net worth of $298 and a transaction amount of +(–$427). Mark explained his solution while writing on the board (see **fig. 4**):

Teacher: Where did he get –$129 from? I need you to try to make sense of this. He starts off, here is the original net worth, and then what will his first jump be? Norman?

Norman: $298.

Teacher: How much debt has he added so far? [Students have difficulty answering this question.] Does Mark start here, all the way down to $427? In one big jump?

Students: No.

Teacher: What is his first jump?

Anthony: Going down to 0.

Teacher: How much?

Anthony: Going down $298.

Teacher: Norman, is that the whole jump he needed to go? No, he is a little bit more. The question is how much is that little bit more. Pretend we did not know what it was. He has already gone down $298. How much does he have to go altogether? [Students say $427.] He had to go $427. He went down a little bit. How much does he have left? He cannot count his fingers, can he? $298, $299. What is the quick way? Sally?

Sally: He subtracts $298.

Teacher: $298 from what?

Students: $427.

Teacher: That is how much he had left to jump. He has to go $129 more. Where is that going to land him?

Students: –$129

Mark's sophisticated solution involved crossing through 0 in two jumps. He was able to see that the domain of integers comprised both positive and negative numbers and that the solution made more sense if one ordered first the positive numbers and then the negative numbers. Most students could not understand how he reached –$129. The teacher wanted the entire class to understand this solution and asked how many jumps Mark made on the number line. They could see the two jumps and that the first jump was in the amount of $298, which took him to 0. The teacher then asked if he was done at $298 or if he had to move again. Students noticed that an additional move was necessary. To find this figure, Sally suggested subtracting the amount from $427. The students were able to see that Mark reached –$129 by moving to 0 and then progressing down the number line by the remaining amount by subtracting $298 from the total.

One essential role of a teacher is to help develop and support students' mathematical reasoning.

To support student understanding during the discussion, the teacher represented the students' statements on the number line. This visual tool helped them understand this sophisticated solution. Instead of explaining each step of Mark's explanation, she asked questions along the way. Thus, during the teacher's reiteration of Mark's solution, the students as a class produced a sophistical solution. Being part of the process helped students understand the solution.

CONCLUSION

Teachers have essential roles in developing students' mathematical ideas. They also need to know how to support students' mathematical reasoning from concrete to abstract; motivate them to make conjectures and prove their conjectures; and give them opportunities to explore efficient, different, and sophisticated solutions. This article gave an overview of how one expert mathematics teacher, who employs NCTM's Standards, used effective teaching practices to achieve these goals. Although the classroom context can be different, these core practices can be used in other areas and continue to be helpful to teachers.

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